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Distributed Controller Design for LPV/LFT Distributed Systems in Input-Output Form ⋆

Qin Liu * **Hossam S. Abbas** ** **Javad Mohammadpour Velni** *
Herbert Werner ***

* *Complex Systems Control Laboratory, College of Engineering, University of Georgia, Athens, GA 30602 USA (e-mail: qin.liu@tuhh.de; javadm@uga.edu)*
** *Electrical Engineering Department, Faculty of Engineering, Assiut University, Assiut, 71515, Egypt (e-mail: hossam.abbas@aun.edu.eg)*
*** *Institute of Control Systems, Hamburg University of Technology, Hamburg, 21073, Germany (e-mail: h.werner@tuhh.de)*

Abstract: This paper considers the controller design problem for parameter-varying distributed systems, whose time/space-varying dynamics can be characterized by temporal/spatial linear parameter-varying (LPV) models defined at the spatially-discretized subsystems. Assuming a rational functional dependence on the scheduling parameters, the distributed LPV model in linear fractional transformation (LFT) form renders analysis and synthesis conditions at the subsystems level with the application of Finsler's lemma and the full block S-procedure technique. The designed distributed controller inherits the interconnected structure of the plant and has a (predefined) fixed structure. Simulation results using a spatially-varying heat equation demonstrate the satisfactory performance of the proposed control design method.

Keywords: Linear parameter-varying (LPV) systems, linear fractional transformation (LFT), distributed systems, input-output form, fixed-structure controller

1. INTRODUCTION

In the last decade, much attention has been paid to address the control problem of large-scale spatially distributed systems, whose distributed dynamics can be captured using partial differential equations (PDEs), see e.g., Jovanovic (2004); Bamieh et al. (2002); D'Andrea and Dullerud (2003); Stewart (2000). Due to the underlying large system order, a centralized controller of this class of systems is often numerically infeasible to solve; moreover, it is sensitive to actuator and sensor failure.

Instead of preserving the continuous nature in space, an effective framework was proposed in D'Andrea and Dullerud (2003), where the overall system can be treated as the interconnection of spatially-discretized subsystems, and each subsystem is equipped with actuating and sensing capabilities. The resulting localized subsystems of much smaller order can be exploited for distributed controller design. If the spatially-discretized subsystems are linear and invariant under temporal and spatial translation, the system is called linear time- and space-invariant (LTSI). However, most distributed systems in practice exhibit linear time- and space-varying (LTSV) properties, due to boundary conditions, non-uniform physical properties, etc. To address the temporal and/or spatial variation of LTSV systems, the linear parameter-varying (LPV) technique, which was first introduced in Shamma (1988) to cope with time-varying systems, can be extended in a systematic way to solve analogous problems of distributed systems. If an LTSV system can be parametrized as functions of temporal- and/or spatial-scheduling parameters, a temporal/spatial LPV model can be used to capture the varying system dynamics over the temporal/spatial scheduling parameter set.

Most research work on distributed control relies on the state-space representation of the plant model, and considers full-order controller design, i.e., controller order is equal or larger than the plant order, see D'Andrea and Dullerud (2003) for distributed control design of LTSI systems, see Wu (2003) and Liu and Werner (2016) for gain-scheduled control of LTSV systems. The importance of controller design in input-output (I-O) form is threefold. Firstly, the experimentally identified models are in I-O form, see Liu and Werner (2013) for the black-box identification of temporal/spatial LPV I-O models. Secondly, it has been demonstrated in Tóth et al. (2012) that when converting an LPV I-O model into its state-space representation, the phenomenon of *dynamic dependence* occurs and requires non-trivial treatment. This phenomenon applies to the state-space realization of temporal/spatial LPV I-O models as well. Thirdly, the I-O framework motivates *fixed-structure* controller design, which has predefined temporal and spatial order and structure. Compared to the state-space full-order controller design, a fixed-structure controller of smaller order can largely reduce the computational complexity. Its benefits become more significant when dealing with large order systems. Recent results on I-O controller design can be found in Wollnack et al. (2013), Wollnack and Werner (2015b), Wilfried et al. (2007), etc.

Inspired by the work in Wollnack and Werner (2015a), where distributed controller design for polytopic LPV systems of **affine** dependence on scheduling parameters has been ad-

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dressed, this paper considers a rather general class of LTSV systems whose coefficients are **rational** functions of scheduling parameters—a case where the polytopic LPV synthesis technique can not be applied. The rational functional dependence leads to the representation of the LPV model in linear fractional transformation (LFT) form–the so-called LPV/LFT model. Of interest here is the distributed controller design that inherits the interconnected structure of the plant, such that analysis and synthesis conditions are developed at the subsystem level—independent of the number of subsystems. Furthermore, the distributed controller can accommodate fixed structure. It is well known that the fixed-structure controller design leads to non-convex synthesis conditions that can be formulated in terms of bilinear matrix inequalities (BMIs). With the application of the full block S-procedure technique introduced in Scherer (2001) and further imposing constraints on the multiplier, it will be shown that it suffices to check a finite number of BMI constraints in the parameter set. Furthermore,

The paper is organized as follows: Section 2 recaps two lemmas. Section 3 presents the multidimensional I-O models that define the interconnected dynamics of both open- and closed-loop LTSV systems at the subsystem level, as well as their LPV/LFT representations. Analysis and synthesis conditions are derived in Section 4. A numerical example is employed in Section 5 to demonstrate the performance of the proposed controller design technique. Conclusions are drawn in Section 6.

2. PRELIMINARIES

This section revisits Finsler's Lemma (see de Oliveira and Skelton (2001)) and the full block S-procedure technique (see Scherer (2001)). They will be employed later to derive the analysis and synthesis conditions proposed in this paper.

Lemma 1. (Finsler's Lemma). Let $x \in \mathbb{R}^n$, $Q \in \mathbb{S}^n$ and $B \in \mathbb{R}^{m\times n}$ such that $\mathrm{rank}(B) < n$. The following statements are equivalent:

i) $x^{\mathrm{T}}Qx < 0, \forall Bx = 0, x \neq 0$.
ii) $\exists X \in \mathbb{R}^{n\times m} : Q + XB + B^{\mathrm{T}}X^{\mathrm{T}} < 0$.

Lemma 2. (Full Block S-Procedure). Given a quadratic matrix inequality

$$\mathcal{R}^{\mathrm{T}}(\Theta)M\mathcal{R}(\Theta) < 0, \tag{1}$$

with the gain-scheduled matrix $\mathcal{R}(\Theta)$ written in a lower LFT form

$$\mathcal{R}(\Theta) = \Theta \star \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix}, \tag{2}$$

and the structured scheduling block $\Theta \in \mathbf{\Theta}$ where $\mathbf{\Theta}$ denotes a compact set and

$$\mathbf{\Theta} = \{\Theta : \mathrm{diag}\{\theta_1 I_{r_1}, \ldots, \theta_{n_\theta} I_{r_{n_\theta}}\}, |\theta_i| < 1, i = 1, \ldots, n_\theta\},$$

where r_i denotes the multiplicity of the scheduling parameters θ_i.

The inequality (1) holds if and only if there exists a full-block multiplier Π such that

$$[*] \left[\begin{array}{c|c} M & 0 \\ \hline 0 & \Pi \end{array}\right] \left[\begin{array}{cc} R_{11} & R_{12} \\ \hline 0 & I \\ R_{21} & R_{22} \end{array}\right] < 0, \tag{3}$$

and for any $\Theta \in \mathbf{\Theta}$

$$[*]\Pi \begin{bmatrix} \Theta \\ I \end{bmatrix} \geq 0. \tag{4}$$

The transpose of a matrix (or a vector) is denoted as $*$.

3. INTERCONNECTED LTSV SYSTEMS

In this paper, we are interested in distributed systems equipped with a spatially distributed array of collocated actuator-sensor pairs. The attachment of actuator-sensor pairs induces spatial discretization, such that the global system can be considered as the interconnection of localized subsystems, each capable of actuating and sensing. For the simplicity of presentation, we consider subsystems of one spatial dimension and with single input and single output. But the framework is rather general and can be applied to cope with systems of multiple spatial dimensions (see D'Andrea and Dullerud (2003)).

3.1 Plant Model in LPV/LFT I-O Form

After the spatial discretization, a parameter-varying distributed system consists of N_s subsystems $\mathcal{G}$, whose dynamics are governed by two-dimensional I-O model (in the sense that all involved signals are functions of time and one-dimensional space)

$$\mathcal{A}(\theta_t, \theta_s, z_t, z_s)y(k, s) = \mathcal{B}(\theta_t, \theta_s, z_t, z_s)u(k, s), \tag{5}$$

where $\theta_t \in \mathbb{R}^{n_{\theta_t}}$, $\theta_s \in \mathbb{R}^{n_{\theta_s}}$ are temporal and spatial scheduling parameters, respectively, $y(k, s) \in \mathbb{R}^{n_y}$ and $u(k, s) \in \mathbb{R}^{n_u}$ are the output and input signals of subsystem s at time instant k, respectively. For the purpose of clarity, we consider subsystems of single input and single output. Furthermore, z_t and z_s are forward temporal and spatial shift operators, respectively, e.g., $z_t^{-1} z_s y(k, s) = y(k-1, s+1)$, $\mathcal{A}$ and $\mathcal{B}$ are gain-scheduled polynomials and defined as

$$\mathcal{A} = 1 + \sum_{i_k=1}^{n_a} \sum_{i_s=-m_a}^{m_a} a_{(i_k,i_s)}(\theta_t, \theta_s) z_t^{-i_k} z_s^{-i_s}, \tag{6a}$$

$$\mathcal{B} = \sum_{j_k=1}^{n_b} \sum_{j_s=-m_b}^{m_b} b_{(j_k,j_s)}(\theta_t, \theta_s) z_t^{-j_k} z_s^{-j_s}, \tag{6b}$$

where n_a, m_a, n_b, and m_b are the indices of time- and space-shifted outputs and inputs, respectively, and $a_{(i_k,i_s)}$, $b_{(j_k,j_s)}$ are the scheduled weighting coefficients.

The implicit representation of (5) is written as

$$\begin{bmatrix} \bar{\mathcal{A}}(\theta_t, \theta_s) & -\bar{\mathcal{B}}(\theta_t, \theta_s) \end{bmatrix} \begin{bmatrix} \xi_y(k, s) \\ \xi_u(k, s) \end{bmatrix} = 0, \tag{7}$$

where $\bar{\mathcal{A}} \in \mathbb{R}^{n_{\xi_y}}$ and $\bar{\mathcal{B}} \in \mathbb{R}^{n_{\xi_u}}$ are the parameter-dependent coefficient vectors of $\mathcal{A}$ and $\mathcal{B}$, respectively, $\xi_y(k, s) \in \mathbb{R}^{n_{\xi_y}}$ and $\xi_u(k, s) \in \mathbb{R}^{n_{\xi_u}}$ are the conformably temporally- and spatially-shifted outputs and inputs, respectively.

We consider the class of systems, where $\mathcal{A}$ and $\mathcal{B}$ have rational dependence on scheduling parameters θ_t and θ_s. By pulling the scheduling parameters out, each subsystem can be represented as the interconnection of an LTSI system G augmented by local feedback with its own temporal and spatial scheduling blocks as shown in Fig. 1. The LFT representation of (7) is given by

$$\begin{bmatrix} \bar{A}_0 & -\bar{B}_0 & \bar{B}_p & 0 \\ \bar{C}_0 & -\bar{D}_0 & \bar{D}_p & -I \end{bmatrix} \begin{bmatrix} \xi_y(k, s) \\ \xi_u(k, s) \\ p(k, s) \\ q(k, s) \end{bmatrix} = 0, \tag{8}$$

where $p(k, s)$ and $q(k, s)$ denote the input and output of the scheduling block, respectively (see (10) below for the definition of the matrices in (8)). Assume that temporal and spatial variations are decoupled, i.e., the spatial properties of subsystems do not change in time. Thus, the scheduling block satisfies

$$\begin{bmatrix} p_t(k,s) \\ p_s(k,s) \end{bmatrix} = \begin{bmatrix} \Theta_t & \\ & \Theta_s \end{bmatrix} \begin{bmatrix} q_t(k,s) \\ q_s(k,s) \end{bmatrix} \quad \text{or} \quad p(k,s) = \Theta q(k,s), \tag{9}$$

with p_t and $q_t \in \mathbb{R}^{n_{\Theta_t}}$, p_s and $q_s \in \mathbb{R}^{n_{\Theta_s}}$, $\Theta_t \in \mathbf{\Theta}_t$ and $\Theta_s \in \mathbf{\Theta}_s$, where q_t and q_s, p_t and p_s are the inputs and outputs of the temporal and spatial uncertainty channels, respectively. Θ_t and Θ_s are the structured temporal and spatial uncertainties of size n_{Θ_t} and n_{Θ_s}, respectively. $\mathbf{\Theta}_t$ and $\mathbf{\Theta}_s$ are two compact sets with the uncertainties structured in diagonal matrices form as

$$\mathbf{\Theta}_t = \{\Theta_t : \operatorname{diag}\{\theta_{t_1} I_{r_{\theta_{t_1}}}, \ldots, \theta_{t_{n_t}} I_{r_{\theta_{t_{n_t}}}}\}, |\theta_{t_i}| < 1, i = 1, \ldots, n_t\}$$

$$\mathbf{\Theta}_s = \{\Theta_s : \operatorname{diag}\{\theta_{s_1} I_{r_{\theta_{s_1}}}, \ldots, \theta_{s_{n_s}} I_{r_{\theta_{s_{n_s}}}}\}, |\theta_{s_i}| < 1, i = 1, \ldots, n_s\},$$

where $r_{\theta_{ti}}$ and $r_{\theta_{si}}$ denote the multiplicity of scheduling parameters θ_{t_i} and θ_{s_i}, respectively.

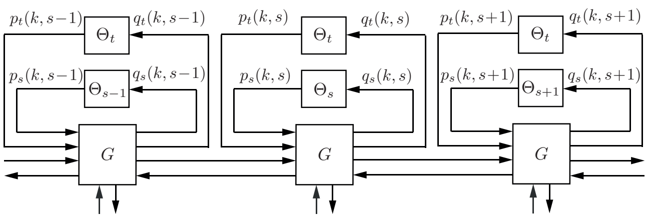

Figure 1. Time-/space-varying interconnected systems in LFT representation.

Assume the well-posedness of the LFT representation (8) and (9), i.e., the existence of $(I - \bar{D}_p\Theta)^{-1}$ for all $\Theta \in \mathbf{\Theta}$. The LPV model (5) can be recovered by applying the lower LFT definition, i.e.,

$$\begin{bmatrix} \bar{\mathcal{A}} & -\bar{\mathcal{B}} \end{bmatrix} = \begin{bmatrix} \bar{A}_0 & -\bar{B}_0 \end{bmatrix} + \bar{B}_p \Theta (I - \bar{D}_p \Theta)^{-1} \begin{bmatrix} \bar{C}_0 & -\bar{D}_0 \end{bmatrix}. \tag{10}$$

Note that in the rest of the paper, a calligraphic uppercase letter denotes a gain-scheduled matrix, whereas a regular uppercase letter denotes a constant matrix.

3.2 Fixed-Structure Controller

A fixed-structure controller design (as opposed to full-order controller) that inherits the interconnected properties of the plant is considered here. The controller dynamics defined at the subsystem level can be expressed as

$$\mathcal{A}^k(\theta_t^k, \theta_s^k, z_t, z_s) u(k,s) + \mathcal{B}^k(\theta_t^k, \theta_s^k, z_t, z_s) e(k,s) = 0. \tag{11}$$

In case of a reference tracking problem, the control error is defined as $e(k,s) = r(k,s) - y(k,s)$, where the reference is given as $r(k,s)$. Controller polynomials $\mathcal{A}^k$ and $\mathcal{B}^k$ are scheduled by controller scheduling parameters θ_t^k and θ_s^k as

$$\mathcal{A}^k = 1 + \sum_{i_k=1}^{n_a^k} \sum_{i_s=-m_a^k}^{m_a^k} a^k_{(i_k,i_s)}(\theta_t^k, \theta_s^k) z_t^{-i_k} z_s^{-i_s}, \tag{12a}$$

$$\mathcal{B}^k = \sum_{j_k=1}^{n_b^k} \sum_{j_s=-m_b^k}^{m_b^k} b^k_{(j_k,j_s)}(\theta_t^k, \theta_s^k) z_t^{-j_k} z_s^{-j_s}. \tag{12b}$$

The controller scheduling block can be chosen as a copy or a function of the plant scheduling block. This issue will be further discussed in Section 4. In this context, fixed-structure means that the polynomial orders n_a^k, m_a^k, n_b^k, and m_b^k, as well as the functional dependence of coefficients a^k and b^k on the scheduling parameters can be predefined by the designer.

By closing the loop between the plant and the controller subsystem, the generalized plant of a controlled subsystem, where the weighting filters W_s and W_{ks} are included to shape sensitivity and control sensitivity of the closed-loop system, respectively, is shown in Fig. 2. Defining the coefficient vectors $\bar{\mathcal{A}}^k$ and $\bar{\mathcal{B}}^k$ of the controller $\mathcal{K}$ and those of the filters W_s and W_{ks} in a similar way as the plant, the LPV implicit representation of the generalized plant takes the form

$$\begin{bmatrix} \bar{\mathcal{A}} & -\bar{\mathcal{B}} & 0 & 0 & 0 \\ \bar{\mathcal{B}}^k & \bar{\mathcal{A}}^k & 0 & 0 & -\bar{\mathcal{B}}^k \\ \bar{B}^s & 0 & \bar{A}^s & 0 & -\bar{B}^s \\ 0 & \bar{B}^{ks} & 0 & -\bar{A}^{ks} & 0 \end{bmatrix} \begin{bmatrix} \xi_y(k,s) \\ \xi_u(k,s) \\ \xi_{z_1}(k,s) \\ \xi_{z_2}(k,s) \\ \xi_w(k,s) \end{bmatrix} := \mathcal{R}(\theta_t, \theta_s, \theta_t^k, \theta_s^k) \xi(k,s) = 0. \tag{13}$$

Note that the weighting filters can in general be parameter-dependent as the plant and the controller. For the sake of presentation simplicity, we consider here only parameter-invariant filters W_s and W_{ks}.

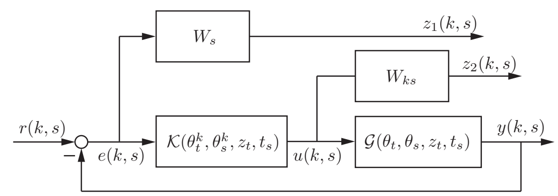

Figure 2. Generalized plant of a controlled subsystem.

By pulling the plant and controller scheduling blocks out as shown in Fig. 3, the generalized plant in LFT form is expressed as

$$\left[\begin{array}{ccccc|cccc} \bar{A}_0 & -\bar{B}_0 & 0 & 0 & 0 & \bar{B}_p & 0 & 0 & 0 \\ \bar{B}_0^k & \bar{A}_0^k & 0 & 0 & -\bar{B}_0^k & 0 & \bar{B}_p^k & 0 & 0 \\ \bar{B}^s & 0 & \bar{A}^s & 0 & -\bar{B}^s & 0 & 0 & 0 & 0 \\ 0 & \bar{B}^{ks} & 0 & -\bar{A}^{ks} & 0 & 0 & 0 & 0 & 0 \\ \bar{C}_0 & -\bar{D}_0 & 0 & 0 & 0 & \bar{D}_p & 0 & -I & 0 \\ \bar{D}_0^k & \bar{C}_0^k & 0 & 0 & -\bar{D}_0^k & 0 & \bar{D}_p^k & 0 & -I \end{array}\right] \left[\begin{array}{c} \xi_y(k,s) \\ \xi_u(k,s) \\ \xi_{z_1}(k,s) \\ \xi_{z_2}(k,s) \\ \xi_r(k,s) \\ \hline p(k,s) \\ p^k(k,s) \\ q(k,s) \\ q^k(k,s) \end{array}\right] = 0, \tag{14}$$

with the scheduling channel partitioned as

$$\left[\begin{array}{c} p(k,s) \\ \hline p^k(k,s) \end{array}\right] = \Upsilon \left[\begin{array}{c} q(k,s) \\ \hline q^k(k,s) \end{array}\right], \tag{15}$$

and $\Upsilon \in \mathbf{\Upsilon}$, where $\Upsilon := \operatorname{diag}\{\Theta_t, \Theta_s, \Theta_t^k, \Theta_s^k\}$, $\mathbf{\Upsilon}$ is an augmented compact set defined as $\mathbf{\Upsilon} := \operatorname{diag}\{\mathbf{\Theta}_t, \mathbf{\Theta}_s, \mathbf{\Theta}_t^k, \mathbf{\Theta}_s^k\}$.

Define a matrix M as

$$M = \left[\begin{array}{c|c} M_{11} & M_{12} \\ \hline M_{21} & M_{22} \end{array}\right] = \left[\begin{array}{ccccc|cc} \bar{A}_0 & -\bar{B}_0 & 0 & 0 & 0 & \bar{B}_p & 0 \\ \bar{B}_0^k & \bar{A}_0^k & 0 & 0 & -\bar{B}_0^k & 0 & \bar{B}_p^k \\ \bar{B}^s & 0 & \bar{A}^s & 0 & -\bar{B}^s & 0 & 0 \\ 0 & \bar{B}^{ks} & 0 & -\bar{A}^{ks} & 0 & 0 & 0 \\ \hline \bar{C}_0 & -\bar{D}_0 & 0 & 0 & 0 & \bar{D}_p & 0 \\ \bar{D}_0^k & \bar{C}_0^k & 0 & 0 & -\bar{D}_0^k & 0 & \bar{D}_p^k \end{array}\right]. \tag{16}$$

The lower LFT computation recovers $\mathcal{R}$ in (13), i.e.,

$$\mathcal{R}(\Upsilon)\xi(k,s) = (\Upsilon \star M)\xi(k,s) = 0, \tag{17}$$

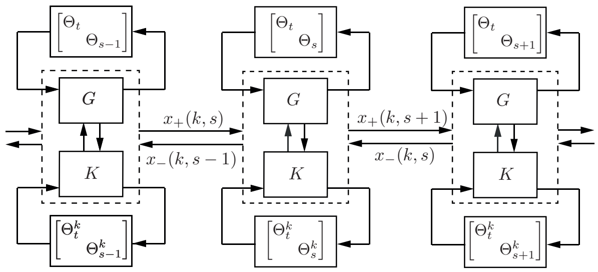

Figure 3. Closed-loop system in LFT form.

where $\star$ denotes the star product of two matrices, i.e., $\Upsilon \star M := M_{11} + M_{12}\Upsilon(I - M_{22}\Upsilon)^{-1}M_{21}$.

In (14), each signal vector $\xi_i(k,s)$ $(i = y, u, z_1, z_2, r)$ can be partitioned into three parts as

$$\xi_i(k,s) = \left[\xi_{i,t}^{\mathrm{T}}(k,s)\ \xi_{i,+}^{\mathrm{T}}(k,s)\ \xi_{i,-}^{\mathrm{T}}(k,s)\right]^{\mathrm{T}}, \qquad (18)$$

where $\xi_{i,t}(k,s)$ denotes the temporal variables, $\xi_{i,+}(k,s)$ and $\xi_{i,-}(k,s)$ denote the positive and negative spatial variables received from neighboring subsystems, respectively. A state-like vector $x(k,s) \in \mathbb{R}^{n_x}$ can then be extracted from $\xi(k,s)$ as

$$x(k,s) = \begin{bmatrix} x_t(k,s) \\ x_+(k,s) \\ x_-(k,s) \end{bmatrix}, \quad \mathbf{\Delta}x(k,s) = \begin{bmatrix} x_t(k+1,s) \\ x_+(k,s+1) \\ x_-(k,s-1) \end{bmatrix},$$

where x_t, x_+ and x_- denote the temporal, the positive and negative spatial states, respectively. Subsystems exchange information through spatial states as shown in Fig. 3. The augmented shift operator $\mathbf{\Delta}$ is defined as $\mathbf{\Delta} = \mathrm{diag}\{z_t, z_s, z_s^{-1}\}$. Selection matrices $\Pi_1 = \mathrm{diag}\{\Pi_{1t}, \Pi_{1+}, \Pi_{1-}\}$ and $\Pi_2 = \mathrm{diag}\{\Pi_{2t}, \Pi_{2+}, \Pi_{2-}\}$ are properly determined such that

$$x(k,s) = \Pi_1\xi(k,s), \quad \mathbf{\Delta}x(k,s) = \Pi_2\xi(k,s).$$

4. CONTROLLER SYNTHESIS CONDITION

In this section, analysis and synthesis conditions defined at the subsystem level are derived. Compared to a centralized control scheme, the computational complexity of distributed controller design is reduced to the size of a single subsystem. With the application of the full block S-procedure, it will be shown that it suffices to check a finite number of constraints within the parameter set.

4.1 Analysis Condition

Quadratic performance of the global system is taken as the performance measure. An asymptotically stable system is said to have quadratic performance γ $(\gamma > 0)$ if

$$\sum_{k=1}^{\infty}\sum_{s=-\infty}^{\infty} [*] \begin{bmatrix} -\gamma I & 0 \\ 0 & \frac{1}{\gamma}I \end{bmatrix} \begin{bmatrix} r(k,s) \\ z(d,s) \end{bmatrix} \leq 0. \qquad (19)$$

The analysis condition that establishes stability and quadratic performance of the closed-loop system is formulated as follows.

Theorem 1. A system in the form of (17) is asymptotically stable and has quadratic performance γ, if there exist a structured matrix P and a gain-scheduled matrix $\mathcal{F}(\Upsilon)$, such that for any $\Upsilon \in \mathbf{\Upsilon}$,

$$P = P^{\mathrm{T}} = \begin{bmatrix} P_t & 0 \\ 0 & P_s \end{bmatrix}, \quad P_t > 0, \quad \det(\mathrm{P_s}) \neq 0, \qquad (20a)$$

$$[*]\left[\begin{array}{cc|cc|cc} P & 0 & & & & \\ 0 & -P & & & & \\ \hline & & I & 0 & & \\ & & 0 & -\gamma^2 I & & \\ \hline & & & & 0 & \mathcal{F}(\Upsilon) \\ & & & & \mathcal{F}^{\mathrm{T}}(\Upsilon) & 0 \end{array}\right]\left[\begin{array}{c} \Pi_3 \\ \Pi_4 \\ \hline \Pi_5 \\ \Pi_6 \\ \hline I \\ \mathcal{R}(\Upsilon) \end{array}\right] < 0, \qquad (20b)$$

with $\Pi_3 = \mathrm{diag}\{\Pi_{2t}, \Pi_{2+}, \Pi_{1-}\}$, $\Pi_4 = \mathrm{diag}\{\Pi_{1t}, \Pi_{1+}, \Pi_{2-}\}$, $\Pi_5\xi(k,s) = \left[z_1^{\mathrm{T}}(k,s)\ z_2^{\mathrm{T}}(k,s)\right]^{\mathrm{T}}$ and $\Pi_6\xi(k,s) = r(k,s)$.

Proof 1. Let the Lyapunov function candidate be chosen as

$$V(k) = \sum_{s=-\infty}^{\infty} x_t^{\mathrm{T}}(k,s)P_t x_t(k,s). \qquad (21)$$

The positive-definiteness of the Lyapunov function $V(k)$ requires $P_t > 0$. Stability of the closed-loop system is assured if the summed energy of the overall system dissipates, i.e.,

$$V(k+1) - V(k) < 0. \qquad (22)$$

It can be easily derived that the summed spatial energy over space cancels out (see Wu (2003)), i.e.,

$$\sum_{s=-\infty}^{\infty}\left([*]\,P_s\begin{bmatrix} x_+(k,s+1) \\ x_-(k,s) \end{bmatrix} - [*]\,P_s\begin{bmatrix} x_+(k,s) \\ x_-(k,s-1) \end{bmatrix}\right) = 0, \qquad (23)$$

where P_s is indefinite due to the spatial non-causality, i.e., P_s has both positive and negative eigenvalues.

Thus, (22) is equivalent to

$$\sum_{s=-\infty}^{\infty}\left([*]\,P\begin{bmatrix} x_t(k+1,s) \\ x_+(k,s+1) \\ x_-(k,s) \end{bmatrix} - [*]\,P\begin{bmatrix} x_t(k,s) \\ x_+(k,s) \\ x_-(k,s-1) \end{bmatrix}\right)$$

$$= \sum_{s=-\infty}^{\infty}\left([*]\begin{bmatrix} P & 0 \\ 0 & -P \end{bmatrix}\begin{bmatrix} \Pi_3 \\ \Pi_4 \end{bmatrix}\xi(k,s)\right) < 0. \qquad (24)$$

After incorporating quadratic performance into the stability by applying the S-procedure, the LMI representation of the bounded real lemma at the subsystem level can be formulated as

$$[*]\left[\begin{array}{cc|cc} P & 0 & & \\ 0 & -P & & \\ \hline & & I & 0 \\ & & 0 & -\gamma^2 I \end{array}\right]\begin{bmatrix} \Pi_3 \\ \Pi_4 \\ \Pi_5 \\ \Pi_6 \end{bmatrix}\xi(k,s) < 0. \qquad (25)$$

Given a fixed value of Υ, the equality (17) and inequality (25) are components of item i) in Finsler's Lemma. Invoking its equivalent statement item ii) implies that there exists a matrix $\mathcal{F}(\Upsilon)$ for each fixed Υ, such that (20b) holds.

4.2 Synthesis Conditions

Although Theorem 1 establishes asymptotic stability and quadratic performance of the closed-loop system, condition (20b) requires checking an infinite number of constraints inside the variation space $\mathbf{\Upsilon}$. Assuming a rational dependence of $\mathcal{F}(\Upsilon)$ on Υ, its quadratic LFT form is written as

$$\mathcal{F}(\Upsilon) = T_F^{\mathrm{T}}(\Upsilon)\hat{F}T_F(\Upsilon) \qquad (26)$$

where $\hat{F}$ denotes a constant matrix. Under the assumption of well-posedness, the gain-scheduled matrix $T_F(\Upsilon)$ can be expressed as

$$T_F(\Upsilon) = T_{F11} + T_{F12}\Upsilon(I - T_{F22}\Upsilon)^{-1}T_{F21}. \qquad (27)$$

The synthesis condition is derived with the application of the full block S-procedure as follows:

Theorem 2. Consider a distributed LPV/LFT plant model in the form of (8) and (9). There exists a fixed-structure controller which guarantees well-posedness, asymptotic stability and quadratic performance γ of the closed-loop system in the form of (14) and (15), if there exist a structured matrix P, a matrix $\hat{F}$, and a multiplier X such that

$$P = P^{\mathrm{T}} = \begin{bmatrix} P_t & 0 \\ 0 & P_s \end{bmatrix}, \quad P_t > 0, \quad \det(\mathrm{P_s}) \neq 0, \tag{28a}$$

$$[*] \left[\begin{array}{cc|cc|cc|c} P & 0 & & & & & \\ 0 & -P & & & & & \\ \hline & & I & 0 & & & \\ & & 0 & -\gamma^2 I & & & \\ \hline & & & & 0 & \hat{F} & \\ & & & & \hat{F} & 0 & \\ \hline & & & & & & X \end{array}\right] \left[\begin{array}{cc} T_{11} & T_{12} \\ \hline 0 & I \\ T_{21} & T_{22} \end{array}\right] < 0, \tag{28b}$$

$$[*]^{\mathrm{T}} X \begin{bmatrix} \Omega \\ I \end{bmatrix} \geq 0, \tag{28c}$$

with Ω defined as $\Omega := \mathrm{diag}\{\Upsilon, \Upsilon, \Upsilon\}$, and

$$T_{11} = S_{11} N_{11}, \quad T_{12} = [S_{12} \;\; S_{11} N_{12}],$$
$$T_{21} = \begin{bmatrix} S_{21} N_{11} \\ N_{21} \end{bmatrix}, \quad T_{22} = \begin{bmatrix} S_{22} & S_{21} N_{12} \\ 0 & N_{22} \end{bmatrix}, \tag{29}$$

where

$$S_{11} = \begin{bmatrix} I & 0 & 0 \\ 0 & T_{F11} & 0 \\ 0 & 0 & T_{F11} \end{bmatrix}, \quad S_{12} = \begin{bmatrix} 0 & 0 \\ T_{F12} & 0 \\ 0 & T_{F12} \end{bmatrix}, \tag{30}$$

$$S_{21} = \begin{bmatrix} 0 & T_{F21} & 0 \\ 0 & 0 & T_{F21} \end{bmatrix}, \quad S_{22} = \begin{bmatrix} T_{F22} & 0 \\ 0 & T_{F22} \end{bmatrix}, \tag{31}$$

$$N_{11} = \left[\Pi_3^{\mathrm{T}} \; \Pi_4^{\mathrm{T}} \,\vdots\, \Pi_5^{\mathrm{T}} \; \Pi_6^{\mathrm{T}} \,\vdots\, I \;\; M_{11}^{\mathrm{T}}\right]^{\mathrm{T}}, \quad N_{21} = M_{21}, \tag{32}$$

$$N_{12} = \left[0 \; 0 \; 0 \; 0 \,\vdots\, 0 \;\; M_{12}^{\mathrm{T}}\right]^{\mathrm{T}}, \quad N_{22} = M_{22}. \tag{33}$$

Proof 2. The proof follows by applying the full Block S-procedure to (20b), and is omitted due to lack of space.

Note that the application of the full block S-procedure separates the augmented scheduling block Υ from (20b), such that the main condition (28b) is parameter independent, whereas the multiplier condition (28c) still involves an infinite number of constraints in the parameter set. Imposing additional constraints on the multiplier X can reduce (28c) into a finite number of constraints at the price of increased conservatism. With $X_{22} > 0$ and the controller scheduling block being a copy of the plant one, i.e., $\Theta_t^k = \Theta_t$ and $\Theta_s^k = \Theta_s$, (28c) is convexified. It is then sufficient to only check (28c) at the vertices of Ω. Furthermore, condition (28c) can be rendered trivially fulfilled when (D, G)-scaling (see Dettori and Scherer (2001)) is used. This is realized by imposing commutability between Ω and components of X, i.e., $X \in \boldsymbol{\mathcal{X}}$, where the multiplier set $\boldsymbol{\mathcal{X}}$ is defined as

$$\boldsymbol{\mathcal{X}} = \left\{ X \in \mathbb{R}^{2n_\Upsilon \times 2n_\Upsilon} \middle| X = X^{\mathrm{T}} = \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^{\mathrm{T}} & X_{22} \end{bmatrix}, \right.$$
$$X_{ij} = \mathrm{diag}\{X_{ij}^t, X_{ij}^s, X_{ij}^{t,k}, X_{ij}^{s,k}\}, X_{ij}^t \in \mathbb{R}^{n_{\Theta_t} \times n_{\Theta_t}},$$
$$X_{ij}^s \in \mathbb{R}^{n_{\Theta_s} \times n_{\Theta_s}}, X_{ij}^{t,k} \in \mathbb{R}^{n_{\Theta_t^k} \times n_{\Theta_t^k}}, X_{ij}^{s,k} \in \mathbb{R}^{n_{\Theta_s^k} \times n_{\Theta_s^k}},$$
$$\left. X_{ij}\Omega = \Omega X_{ij}, i, j = 1, 2, X_{11} = -X_{22} < 0, X_{12}^{\mathrm{T}} = -X_{12} \right\}.$$

With the use of (D, G)-scaling, under the assumption that $|\theta_{ti}| < 1$ and $|\theta_{si}| < 1$, the multiplier condition (28c) is then trivially fulfilled. It suffices to consider only (28a) and (28b) to solve for the desired controller.

Fixed-structure controller design results in BMI constraints (28b). A DK iteration based approach has demonstrated its efficiency in Wollnack and Werner (2015a) and can be applied:

1. Given a stabilizing controller (via initialization at the first iteration, or found in Step 2), optimize γ over P, $\hat{F}$ and X;
2. Given P, $\hat{F}$ and X found in Step 1, optimize γ over the controller parameters;
3. Go back to Step 1 until γ is minimized.

5. NUMERICAL EXAMPLE

To demonstrate the performance of the proposed distributed control design method, the heat equation of one spatial dimension in Wollnack and Werner (2015a) is taken as a numerical example, i.e.,

$$\frac{\partial y(k,s)}{\partial t} - \kappa(\theta_s) \frac{\partial^2 y(k,s)}{\partial x^2} = u(t,s), \tag{34}$$

where $y(k, s)$ denotes the temperature, and $u(k, s)$ denotes the control input. The thermal diffusivity $\kappa(\theta_s)$ varies with respect to spatial scheduling parameter θ_s. The application of the finite difference method to (34) discretizes the PDE both in time and space. Let the sampling time be chosen as $T_t = 0.01$ s and the overall system of length 10 m be spatially discretized into 41 subsystems, i.e., $s = \{-20, \ldots, 20\}$. The resulting sampling space is then $T_s = 0.25$ m. The scheduling parameter θ_s is scheduled by the spatial variable s

$$\theta_s = \big(\tanh(\tfrac{3}{4}s - 10) + \tanh(-\tfrac{3}{4}s - 10) + 2\big)/2. \tag{35}$$

The thermal diffusivity has a rational functional dependence on θ_s, i.e.,

$$\kappa(\theta_s) = \theta_s + \theta_s^2. \tag{36}$$

The LFT formulation of the plant subsystem model gives

$$\bar{A}_0 = [1 \; -1 \; 0 \; 0], \bar{B}_0 = [0 \; T_t \; 0 \; 0], \bar{B}_p = [1 \; 1],$$
$$\bar{C}_0 = \begin{bmatrix} 0 & 2\alpha & -\alpha & -\alpha \\ 0 & 0 & 0 & 0 \end{bmatrix}, \bar{D}_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \bar{D}_p = [0 \; 0], \tag{37}$$

with $\alpha = \frac{T_t}{T_s^2}$ and $\Theta_s = \mathrm{diag}\{\theta_s, \theta_s\}$.

The distributed controller is assumed to have the following predefined structure

$$\bar{A}^k(\theta_s) = \left[1 \;\; a_{(1,0)}^k(\theta_s) \;\; a_{(1,-1)}^k(\theta_s) \;\; a_{(1,1)}^k(\theta_s)\right],$$
$$\bar{B}^k(\theta_s) = \left[b_{(0,0)}^k(\theta_s) \;\; b_{(1,0)}^k(\theta_s) \;\; b_{(1,-1)}^k(\theta_s) \;\; b_{(1,1)}^k(\theta_s)\right].$$

A fixed-structure distributed controller of affine dependence on θ_s is considered here, e.g., $a_{(1,0)}^k(\theta_s) = a_{(1,0)}^{k,0} + \theta_s a_{(1,0)}^{k,1}$, where $a_{(1,0)}^{k,0}$ and $a_{(1,0)}^{k,1}$ are unknown constants and needed to be determined in controller design. Its LFT representation is written as

$$\bar{A}_0^k = \left[1 \;\; a_{(1,0)}^{k,0} \;\; a_{(1,0)}^{k,0} \;\; a_{(1,0)}^{k,0}\right], \bar{B}_0^k = \left[b_{(0,0)}^{k,0} \;\; b_{(1,0)}^{k,0} \;\; b_{(1,0)}^{k,0} \;\; b_{(1,0)}^{k,0}\right],$$
$$\bar{C}_0^k = -\left[0 \;\; a_{(1,0)}^{k,1} \;\; a_{(1,0)}^{k,1} \;\; a_{(1,0)}^{k,1}\right], \bar{B}_p^k = -1, \bar{D}_p^k = 0,$$
$$\bar{D}_0^k = -\left[b_{(0,0)}^{k,1} \;\; b_{(1,0)}^{k,1} \;\; b_{(1,0)}^{k,1} \;\; b_{(1,0)}^{k,1}\right],$$

and $\Theta_s^k = \theta_s$, i.e., the controller scheduling block is part of the plant scheduling block.

Further assume that the matrix $F(\Upsilon)$ is affine in Υ with $\Upsilon = \text{diag}\{\Theta_s, \Theta_s^k\}$, i.e., $F(\Upsilon) = F_0 + \Upsilon F_1$. Its quadratic LFT representation in the form of (26) is given by

$$\hat{F} = \begin{bmatrix} 0 & F_1/2 \\ F_1/2 & F_0 \end{bmatrix}, \quad \begin{bmatrix} T_{F11} & T_{F12} \\ T_{F21} & T_{F22} \end{bmatrix} = \begin{bmatrix} 0 & I \\ I & 0 \\ I & 0 \end{bmatrix}.$$

Moreover, (D, G)-scaling has been imposed on the multiplier.

The performance of the proposed controller design approach is evaluated by tracking given reference inputs. Between 0.1 s and 0.3 s, a unit step is given as the reference at subsystems 18 to 23, whereas between 0.5 s and 0.7 s, the reference is switched to -2. Fig. 4 shows a comparison between the given reference and the closed-loop responses at subsystem 21, whereas Fig. 5 shows the 3-D plot of the closed-loop response and the control input. It can be seen that the controller demonstrates a satisfactory performance in terms of reference tracking.

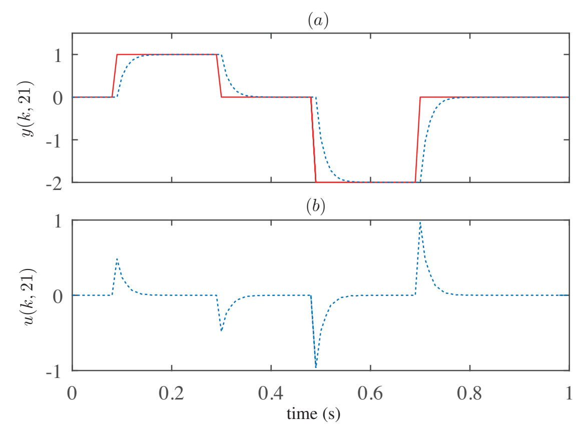

Figure 4. (a): comparison between the reference (red solid) and closed-loop responses (blue dashed) at the subsystem 21. (b): the control input at the subsystem 21.

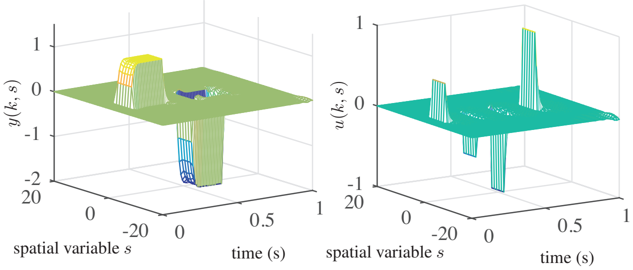

Figure 5. 3-D plot of the closed-loop response (left) and the control input (right).

6. CONCLUSION

In this work, a fixed-structure distributed controller design approach for parameter-varying distributed systems has been developed. We have shown that provided a rational dependence on the scheduling parameters, the implicit I-O representation which describes the localized dynamics at the subsystems can be characterized in LPV/LFT form and leads to subsystem-based synthesis conditions. The designed controller has been evaluated on a spatially-varying heat equation. Stability and a satisfactory performance in terms of reference tracking have been achieved.

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