Robust non-linear control design for systems governed by Burgers' equation subjected to parameter variation

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Abstract: In this study, a robust sliding mode controller is proposed for dynamic systems governed by Burgers' equation with Neumann boundary conditions in the presence of parameter variations. The main objective is to design a reduced-order model based controller at a nominal value of the system parameter that stabilises the full-order model while being robust with respect to model uncertainties in the obtained reduced-order model. The model uncertainties resulted from the variation of the parameter \( \nu \) are discussed under two categories; first, the error arising from the change in the states of the full-order model, and second, the error associated with the estimated proper orthogonal decomposition basis functions at the nominal value of \( \nu \). In this work, the boundedness of the error functions is studied and an estimation of these bounds is obtained in terms of the reduced-order model and matrices of the full-order system that are known \textit{a priori}. Next, the bounds on the error functions are used to design a reduced-order sliding mode controller that guarantees the stability of the full-order model obtained via a finite element approximation of the Burgers' equation for a trajectory tracking problem.

1 Introduction

Computational modelling, simulation, and control of non-linear turbulent systems is a challenging task due to the complexity of the underlying fluid mechanics. There is a handful of research targeting the control design problem for linear, semi-linear and non-linear parabolic, and hyperbolic partial differential equations (PDEs). In [1], stabilisation of a semi-linear parabolic PDE, in which the heat source depends on the temperature of the whole space, is considered by using boundary control. The non-linear boundary control laws are derived to achieve global asymptotic stability. The adaptive boundary stabilisation and control has been investigated for a class of systems described by first-order hyperbolic PDEs with an unknown spatially-varying parameter in [2].

The Navier–Stokes equation describes many of the underlying phenomena in fluid mechanics. This equation is simplified to the Burgers' equation when the flow is considered to be incompressible and with the pressure term removed. The Burgers' equation can be viewed as an intermediate step to capture very critical non-linear convective behaviours that can model shock waves, some boundary layer problems and traffic flow problems among many others [3, 4].

There has been a great deal of research on stabilisation and control of Burgers' equation. Adaptive and non-adaptive stabilisations of the generalised Burgers' equation using non-linear boundary control are studied in [5], and it is shown that the controlled system is exponentially stable in \( L^2 \) for the non-adaptive case. A new approach is also proposed to show the \( L^2 \) regulation of the solution of the generalised Burgers' equation. In [6], a boundary control law and adaptation law for the Burgers' equation with unknown viscosity are proposed without acquiring a priori knowledge of a lower bound on viscosity. This decentralised control law is implemented without the need for a central computer.

Due to their complexity and high computational cost for control-oriented applications, there have been attempts to implement model reduction methods that can result in more suitable models for control purposes. The proper orthogonal decomposition (POD), which is an efficient tool for model reduction, extracts a number of basis functions that would be used in formulating Galerkin projection in a collocation form resulting in a system with finite dimensions with a low number of degrees of freedom [7, 8].

In [9], a reduced-order modelling approach suitable for active control of fluid dynamical systems is proposed based on the POD. The main objective is to reduce the computational burden for numerical simulation of the Navier–Stokes equations for the purpose of optimisation and control of unsteady flows. The possibility of obtaining reduced-order models is examined to potentially reduce computational complexity associated with the Navier–Stokes equation while capturing the essential dynamics using the POD method. POD-based, open-loop and closed-loop optimal control schemes are proposed for the Burgers' equation in [10]. For closed-loop control, suboptimal state feedback strategies are presented and compared to the full-order, model-based controller. In [11], a control design method is proposed based on the use of POD combined with efficient numerical methods for solving the resulting low-order evolutionary Hamilton–Jacobi–Bellman equation. An optimal feedback control for the Burgers' equation is used to evaluate the viability of the proposed methodology.

A recursive model reduction approach, known as adaptive proper orthogonal decomposition (APOD), is implemented in [12, 13]. In [12], APOD is used to initiate and recursively modify locally valid reduced-order models by finding the dominant dynamic behaviour of the underlying physico-chemical systems. A robust state feedback controller is combined with an APOD-based non-linear Luenberger-type switching dynamic observer that decreases required sensor measurements. In [13], an output feedback controller for distributed processes is proposed where infinite-dimensional representation can be decomposed into a slow finite dimensional and fast infinite-dimensional subsystems. APOD is used as a refined ensembling approach to recursively update the eigenfunctions as the closed-loop process evolves through different regions of the state space based on maximising retained information that is received from the infrequent distributed sensor measurements.

The generated POD basis functions solving the underlying eigenvalue problem are dependent on a set of parameters that may not give an accurate estimation of the full-order model associated with a different set of parameters [14, 15]. Hence, the use of reduced-order models can introduce a source of uncertainty imposed by the order of the reduced model. In addition to this,
there are other types of uncertainties, e.g. varying parameters, that might affect the accuracy of the extracted reduced model [2, 16, 17]. Hence, the objective is to obtain an accurate reduced-order representation of the original system while ensuring robustness to uncertainties. The elements of this dictionary are solutions computed for varying values of time and the associated parameter. In [18], a sensitivity analysis is carried out to include the flow and shape parameters influenced during the basis selection process to develop more robust reduced-order models for varying viscosity, changing orientation and shape definition of bodies.

Among various control schemes, the sliding mode is considered as a powerful non-linear control method especially when coping with uncertainties. In [19], the error dynamics regulation between the governing PDE and a target PDE is used to address dynamic shaping problem describing the desired spatio-temporal behaviour. A reduced-order model is extracted and used to obtain the difference between governing and the target reduced-order model that gives the reduced offset dynamics error. Then, an output feedback sliding mode controller is designed to stabilise the reduced error dynamics and correspondingly represent the system and the target spatio-temporal behaviours. In [20], the authors have designed and implemented a non-linear controller for fluid system. However, the designed controller does not take the uncertainties into account either in controller design or in the extracted reduced-order model. This work tackles this issue under various types of model and parametric uncertainties.

In this paper, the model uncertainties are classified into two categories and a robust non-linear controller is proposed for trajectory tracking. First, the model uncertainties arising from the approximation of the full-order model by the reduced-order one are investigated. Furthermore, the error associated with the varying parameter is studied when the POD basis functions extracted at the nominal parameter \( \nu_0 \) are used to estimate the full-order model at the new value of the viscosity \( \nu \). A non-linear controller is designed based on reduced-order sliding mode control (SMC) that is capable of handling various types of uncertainties including parametric and modelling imprecisions.

The advantages of the presented approach are twofold. First, the obtained reduced model is calculated once and the model uncertainties are bounded with the reduced and full-order models in the nominal viscosity \( \nu_0 \). In other words, instead of costly calculation of the basis functions associated with different values of the parameter, the reduced-order model is computed once and an accurate estimation of the associated model uncertainties is obtained. Second, a robust controller is designed taking into account the bounds on uncertainties to capture an uncertain reduced model at the nominal parameter to ensure the desired tracking of the reference trajectory for the full-order model at any viscosity around \( \nu_0 \). The results indicate that the controller designed based on the developed reduced-order model can provide a desirable performance in a reference tracking problem that in turn results in a more efficient control of complex models like Burgers’ equation.

In this paper, \( \ldots \) denotes the inner product of the basis functions and represents the spatial integration of the product of the given basis functions. Also, the \( i \)th Fourier coefficient of the reduced-order model of order \( l \) is shown by \( \mathbf{B}_l^i \) and the \( m \)-dimensional Euclidean space is shown by \( \mathbb{R}^m \). Moreover, Kronecker delta, \( \delta_{ij} \), returns zero for \( i \neq j \) and 1 for \( i = j \). Finally, Hadamard product of the matrices \( A \) and \( B \) is indicated by \( A \odot B \) such that \( [A \odot B]_{ij} = [A]_{ij}[B]_{ij} \).

The remaining of the paper is organised as follows. Section 2 describes the reduce-order modelling of the Burger’s equation and its underlying formulation using POD. The error estimates and the near-parametric reduced-order model based sliding mode controller design for a reference tracking problem are introduced in Section 3. Section 4 gives the simulation results, and finally, Section 5 provides the concluding remarks.

2 Finite element (FE) modelling and POB-based reduced-order modelling of the Burger’s equation

The Burger’s equation can describe turbulence and other complex phenomena in fluid systems. The Burgers’ PDE with Neumann boundary conditions is used to obtain the reduced-order model for control design purposes. Suppose that \( \Omega \) represents the spatial interval \((0, L)\) and that for \( T > 0 \) we define \( Q = (0, T) \times \Omega \) where \( \Omega \) is the spatial interval \((0, L)\). The viscous Burgers’ equation with the viscosity of \( \nu \) and the initial and boundary conditions are described by

\[
\frac{\partial w(t,x)}{\partial t} + \nu \frac{\partial w(t,x)}{\partial x} = \frac{\partial^2 w(t,x)}{\partial x^2} = f(t,x), \quad (1a)
\]

I.C.: \( w_0(0,x) = w_0(x) \),

B.C.: \( w_0(t,0) = u_0(t), \quad w_0(t,L) = u_0(t) \),

\[
(1c)
\]

where \( w(t,x) \) is the fluid velocity with \((t,x) \in Q \). The flux boundary conditions \( u_0(t) \) and \( u_0(t) \) are the varying boundary conditions (i.e. controlled inputs). The spatial derivative of \( w_0 \) is shown by \( w_0 \). The physical interpretation of the various terms can be found in [21], where the Burger’s equation is studied from fluid dynamic perspective as the primary application.

The viscosity in (1a) is defined as \( \nu = 1/Re \), where \( Re \) denotes the Reynolds’ number. The function \( f \) in (1a) is the force term assumed to be square integrable in space and time. We define the Hilbert space of Lebesgue square integrable functions as \( H = L^2(\Omega) \). The function \( f \) is said to be in \( H \) if it satisfies

\[
\int_0^T \| f(t,x) \|^2_H \, dt < \infty.
\]

where \( H = L^2(\Omega) \).

2.1 State-space representation of Burgers’ PDE via FE method

The non-linear state-space representation is obtained via implementing the FE modelling (FEM) method combined with the weak solution approach [22]. Let \( V = H^1(\Omega) \) be the associated Sobolev space as introduced in [23] and define the set of square integrable functions ing to the associated Banach space as \( v \in L^2(0,T;V) \) and \( v \in L^2(0,T;V) \), where \( v \) represents the time derivative of \( v \). Furthermore, assuming that the given initial condition \( w_0(x) \) and the forcing term \( f(t,x) \) belongs to the space of essentially bounded functions, i.e. \( w_0(x) \in L^\infty(\Omega) \) and \( f(t,x) \in L^2(\Omega) \), we introduce the Banach space \( \mathcal{P} = L^2(0,T;V) \cap L^\infty(\Omega) \). The boundedness of the initial conditions and forcing terms in the given spaces are essential in guaranteeing the existence of the weak solution in the Banach space. Therefore, the weak solution satisfies \( w(t,x) \in \mathcal{P} \).

The spatial domain is divided into \( N \) subintervals as \([x_i, x_{i+1}]\). We define \( x_i = x_{i+1} - x_i \) and assume \( h_i = \cdots = h_N = h \). Next, we use the same basis as given in [24, 25]. The velocity of the flow \( w_0(x) \) is approximated by formulating it in the space of piecewise linear basis functions as

\[
w_i(t,x) = \sum_{j=0}^{N} \mathbf{W}_e, i(t) \cdot N_i(x),
\]

where \( \mathbf{W}_e, i(t) \) is the nodal value at the \( i \)th node and time \( t \), i.e. \( w_i(t,x) \). Also, \( N_i(x) \) represents the \( i \)th basis function.

Lemma 1: Assume that there exists a weak solution to (1). Then, a space-state representation of (1) enforcing initial and Neumann boundary conditions for the given input vector \( u(t) = [u(t) \ u(t)]^T \) and Fourier coefficients \( \mathbf{W}_e, i(t) = [\mathbf{W}_e, i(t) \cdots \mathbf{W}_e, i(t)]^T \) is determined to be
\[ W_j(t) = M^{-1}AW_j(t) + M^{-1}b(t, W_j(t), U(t)), \]  
\[ \text{where} \]
\[ A = -\nu S, \quad b(t, W_j(t), U(t)) = -\frac{1}{2}\kappa(K(W_j(t) \ast W_j(t))) + F(t) + \nu LU(t), \]
with
\[ L = \begin{bmatrix} -1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix}_{\nu \times \nu}, \]
\[ [M]_{ij} = \langle N(x), N'(x) \rangle, \quad [S]_{ij} = \langle N(x), N(x) \rangle, \quad F_i(t) = \langle f(t, x), N(x) \rangle, \quad [K]_{ij} = \langle N(x), N(x) \rangle, \]
and with the initial condition represented in the matrix form as
\[ M W_j(0) = J, \]
where \( J = \{w_0(x), N_j(x)\}. \)

**Proof:** The given Galerkin approximation of \( w_j(t, x) \), i.e. equation (2), belongs to the Banach space of the weak solutions \( L^0(0, T; V) \cap L^\infty(0, T) \). Also, the arbitrary and piecewise linear function \( v(x) \) is substituted by the piecewise linear basis function of the FEM, \( N_j(x) \), for \( j = 0, 1, \ldots, N \). To cope with the non-linear term, the group FE (GFE) method is used, which is an alternative approach for FE for solving non-linear elliptic, parabolic, and hyperbolic problems [26, 27]. The non-linear term is represented using GFE in the following form:
\[ w_j^2(t, x) = \sum_{i=0}^{N} W_i^2(t), V_i(x), \]
\[ (N(W_j(t)))_j = \frac{1}{2} \sum_{i=0}^{N} W_i^2(t) \int_0^L N_i(x) N_j(x) \text{d}x. \]
Defining
\[ [K]_{ij} = \langle N(x), N(x) \rangle \]
yields the following quadratic form:
\[ N(W_j(t)) = \frac{1}{2} K(W_j(t) \ast W_j(t)). \]
The Galerkin estimation along with substituting (6) into the weak solution representation in (2) gives
\[ \frac{N}{L} \sum_{i=0}^{N} W_i(t) \int_0^L N_i(x) N_j(x) \text{d}x + \frac{1}{2} \sum_{i=0}^{N} W_i^2(t) \int_0^L N_i(x) N_j(x) \text{d}x - \nu \sum_{i=0}^{N} W_i(t) \int_0^L N_i(x) N_j(x) \text{d}x \]
\[ = \int_0^L f(t, x) N_j(x) \text{d}x. \]
We consider the notations defined in (4) to form the matrix representation of the previous equation
\[ \frac{N}{L} \sum_{i=0}^{N} W_i(t) \int_0^L N_i(x) N_j(x) \text{d}x + \frac{1}{2} K(W_j(t) \ast W_j(t)) - \nu LU(t) = F(t). \]
Rewriting this equation results in the space-space representation (3). We rewrite the initial conditions as
\[ w_j(0, x) = \sum_{i=0}^{N} W_i(0), N_j(x). \]

By enforcing the initial condition (1b) and multiplying it by the test function \( N_j(x) \) from both sides, we have
\[ \sum_{i=0}^{N} W_i(0) \int_0^L N_i(x) N_j(x) = \int_0^L w_j(0, x) N_j(x), \quad j = 0, \ldots, N. \]
The matrix form of this equation is obtained as (5) which can be solved to obtain the initial conditions. \( \square \)

### 2.2 Reduced-order model of the Burgers’ PDE using POD

The POD method is implemented for the model reduction. The details of the approach are given in [25] where it is shown that the model reduction of complex PDE models like the Burgers equation results in a lesser computational burden. The real-valued data matrix \( \mathbf{W}_k = [\mathbf{W}_{k,0}, \ldots, \mathbf{W}_{k,0}\cdots 0\cdots 0] \) consists of \( n \) time snapshots of \( N + 1 \) points in space obtained for parameter \( \nu \), where for \( j \)th time snapshot \( t_j \), we have \( \mathbf{W}_{k, j} = [\mathbf{W}_{k, j}(t_0), \mathbf{W}_{k, j}(t_1), \ldots, \mathbf{W}_{k, j}(t_N)] \). A linear representation of columns of \( \mathbf{W}_{k, j} \) is obtained by finding an optimum set of bases of rank \( l \) via the solution to the following optimisation problem:
\[ J = \int_0^T \left\| \mathbf{W}_k(t) - \sum_{i=1}^{l} \langle \mathbf{W}_k(t), \psi_i(t) \rangle \psi_i(t) \right\|_M^2 \text{d}t \]
\[ \text{s.t. } \langle \psi_i, \psi_j \rangle_M = \delta_{ij} \quad \text{for } 1 \leq i, j \leq l, \]
where \( \{\psi_i\}_{i=1}^{l} \) are set of basis functions and projection operator associated with the space spanned by \( \{\psi_i\}_{i=1}^{l} \) can be defined as
\[ \mathbf{P}_l \mathbf{W}_k(t) = \sum_{i=1}^{l} \langle \mathbf{W}_k(t), \psi_i(t) \rangle \psi_i(t). \]

Next, we set \( \tilde{Y} = \mathbf{M}^{1/2} Y D^{1/2} \in \mathbb{R}^{N+1 \times n} \) that leads to the following eigenvalue formulation that gives the solution to (10):
\[ \tilde{Y} \tilde{Y}^T \psi_i = \lambda_i \psi_i \quad 1 \leq i \leq l, \]
\[ \langle \psi_i, \psi_j \rangle_{\mathbb{R}^{n+1}} = \delta_{ij} \quad 1 \leq i, j \leq l, \]
where \( D = \text{diag}(\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^{n \times n} \) and \( \alpha_i \)'s are trapezoidal coefficients defined by
\[ \alpha_i = \frac{\Delta t}{2}, \quad \alpha_j = \Delta t, \quad \alpha_n = \frac{\Delta t}{2}, \]
with \( j = 2, \ldots, n - 1 \) and \( \Delta t = t_j - t_{j-1} \) [28]. The proper number of basis functions is chosen via calculating the captured energy [29]
\[ \mathcal{E}(l) = \sum_{i=1}^{l} \frac{1}{\mu_i}, \]
where \( d = \text{rank}(\tilde{Y}) \). The solution to Burgers’ equation (flow velocity) is described by the following weighted summation on the reduced-order space:
\[ w_l(t, x) = \sum_{i=1}^{l} \mathbf{W}_{l,i}(t) \psi_i(x), \]
where \( \mathbf{W}_{l,i}, 1 \leq i \leq l, \) are obtained as
\[ \mathbf{W}_{l,i}(t) = \{w_l(t, x), \psi_i(x)\}_M. \]
Remark 1: As proposed in [25], the reduced-order model satisfying the given initial condition and Neumann boundary conditions are determined in state-space form as

$$
\dot{\mathbf{y}}(t) = A_l \mathbf{y}(t) + \mathbf{g}_l(t, \mathbf{y}(t), U(t)),
$$

where $A_l = -\nu L_l$ and (see equation below), with

$$
M_l = I, \quad F_l(t) = (\Psi_l)^T F(t), \quad L_l = (\Psi_l)^T L,
$$

$$
S_l = (\Psi_l)^T S \Psi_l,
$$

$$
N_l(\mathbf{y}(t)) = \frac{1}{2}(\Psi_l)^T K(\Psi_l) \Psi_l(t) + (\Psi_l)^T \mathbf{F}_l(t).
$$

\section{Error estimates and SMC design using reduced-order model}

Designing complex controllers for real-time applications should consider their computational burden in order to ensure practicality of the proposed controller. To tackle this issue, model reduction methods provide important tools that are used to capture the main dynamics of the system while representing the original model in a lower-order model form. A sliding mode controller is designed based on the reduced-order model of Burger's PDE where it accounts for the varying viscosity. The reduced-order model is obtained at a certain viscosity and the designed controller is expected to stabilise the system and control the system with a viscosity of $\nu$.

\subsection{3.1 Preliminaries for SMC design}

The SMC is based on the definition of a surface that is introduced using the system and desired outputs. The associated full-order model is obtained as

$$
\dot{X}(t) = AX(t) + b(t, X(t), U(t)),
$$

$$
Y(t) = CX(t).
$$

For a nominal value of the system parameter, i.e. the viscosity $\nu_0$, the set of basis functions $\Psi_{i_0}^l$ is used to obtain the nominal reduced-order model as

$$
\dot{\mathbf{y}}_{i_0}(t) = A_{i_0}^l \mathbf{y}_{i_0}(t) + \mathbf{g}_{i_0}(t, \mathbf{y}_{i_0}(t), U(t)),
$$

$$
Y_{i_0}(t) = C_{i_0}^l \mathbf{y}_{i_0}(t),
$$

and we have

$$
C_{i_0}^l = C \Psi_{i_0}^l.
$$

The control objective is to track a reference trajectory via a sliding surface based on reduced-order model designed in a priori knew the nominal value of the system parameter $\nu_0$. The idea here is to design a reduced-order model based on control law associated with the nominal value $\nu_0$ that ensures the desired reference tracking for the full-order model while the parameter $\nu$ can change. A sliding surface is introduced to ensure that the output tracks the defined reference signal. A surface is defined via the dynamics of the main model where the control input is imposed by the dynamics obtained from the reduced-order model. Implementing the full-order model, the following surface is defined:

$$
S(t) = Y(t) - r(t) = C\mathbf{y}(t) - r(t),
$$

where the surface is represented as $S(t) = [S_1(t), S_2(t)]^T$, $Y_1(t) = [y_{11}(t), y_{12}(t)]^T$, and the reference signal is defined as $r(t) = [r_1(t), r_2(t)]^T$. The reduced-order sliding surface is defined as

$$
S_l(t) = CM_{i_0}^l \mathbf{y}_{i_0}(t) + C_{i_0}^l S_l(t) - r(t),
$$

where $S_l(t) = \mathbf{y}(t) - \Psi_{i_0}^l \mathbf{y}_{i_0}(t)$. Due to the need for the states of the reduced model for the controller design, a reduced-order observer is designed as proposed in [30]. The details of the observer design are presented later in this paper. The sliding mode controller is constructed from switching and equivalent control laws. The switching part of the control law that is designed with respect to the definition of reduced-order surface drives the states of the system in the direction of a previously defined surface in the presence of model uncertainties. It will be shown that the switching control law that implements the dynamics on the reduced-order surface effectively stabilises the original system. In addition, the equivalent control law ensures that the states of the system remain on the defined surface. The control law is represented as

$$
U(t) = u_{eq}(t) + u_{os}(t),
$$

where the first term is equivalent control law and the second one is the switching control law. The equivalent control law is activated when states are on the sliding surface. Hence, we have

$$
S(t) = Y(t) - r(t) = C\mathbf{y}(t) - r(t).
$$

Since we do not have online access to the full-order model states or varying viscosity $\nu$, the dynamics of the sliding surface is defined in terms of the states of the reduced-order model as

$$
\dot{S}_l = C_{i_0}^l \mathbf{y}_{i_0}(t) + (C_{i_0}^l \mathbf{y}_{i_0}(t) - C_{i_0}^l \mathbf{y}_{i_0}(t)) - r(t)
$$

$$
= CA_{i_0}^l \mathbf{y}_{i_0}(t) + \Delta^l_{i_0} S_l
$$

$$
= \frac{1}{2}CM_{i_0}^l K(\Psi_{i_0}^l) (\Psi_{i_0}^l)^T + \Delta^l_{i_0} f + CM_{i_0}^l F(t) + \nu CM_{i_0}^l L U(t) - r(t),
$$

where the terms representing uncertainties are

$$
\Delta^l_{i_0} f = f (\mathbf{y}(t)) - f (\Psi_{i_0}^l \mathbf{y}_{i_0}(t))
$$

$$
= -\frac{1}{2}CM_{i_0}^l K(\mathbf{y}(t) - \mathbf{y}_l(t)) - (\Psi_{i_0}^l)^T \mathbf{F}_l(t) - (\Psi_{i_0}^l)^T \mathbf{F}_l(t).
$$

The non-linear term described by (28) is assumed to be locally Lipschitz with respect to $\Psi_{i_0}^l \mathbf{y}_{i_0}$ and $\mathbf{y}_l$ in a region $\mathcal{D}$, i.e. for any $\mathbf{y}_l(t), \Psi_{i_0}^l \mathbf{y}_{i_0}(t), \mathbf{y} \in \mathcal{D}$ [30]

$$
\|f(\mathbf{y}_l) - f(\Psi_{i_0}^l \mathbf{y}_{i_0})\| \leq \gamma_h \|\mathbf{y}_l - \Psi_{i_0}^l \mathbf{y}_{i_0}\|,
$$

where $\| \cdot \|$ represents the 2-norm and $\gamma_h$ is the non-negative Lipschitz constant. The locally Lipschitz condition is the central condition to guarantee the existence and uniqueness of the solution to an initial value problem [30]. Hence, two model uncertainty
terms are represented in terms of the states of the full-order and reduced-order models.

3.2 Error estimates for the varying viscosity

The uncertainty terms inherently result from the sensitivity of the POD method with respect to the variation in the system parameter \( \nu \). The main challenge is to represent the confidence region as a function of the changing parameter. The following analysis is carried out to present a priori estimate for the squared error in the Hilbert space for the reference set of POD basis functions \( \Psi_{\nu i} \). The error caused by any variation in the parameter value \( \nu \) can be written as

\[
\| W_0 - \Psi_{\nu i} W_{\nu i} \|_M = \| W_0 - W_\nu + \Psi_{\nu i} - \Psi_{\nu i} W_{\nu i} + \Psi_{\nu i} W_{\nu i} - \Psi_{\nu i} W_{\nu i} \|_M .
\]

After applying Cauchy–Schwarz inequality multiple times, we obtain

\[
\| W_0 - \Psi_{\nu i} W_{\nu i} \|_M \leq 2 \| W_0 - W_\nu \|_M + 4 \| W_0 - \Psi_{\nu i} W_{\nu i} \|_M^2 + 4 \| \Psi_{\nu i} W_{\nu i} - \Psi_{\nu i} W_{\nu i} \|_M^2.
\]

As seen from the right hand side of the inequality above, the boundedness of each error term needs to be investigated to essentially evaluate the bound on \( \| W_0 - \Psi_{\nu i} W_{\nu i} \|_M \).

**Lemma 2:** The estimation error representing the variation in the states of the full-order model associated with changing \( \nu \) in a given time interval \((0, T)\) is bounded by

\[
\| W_0 - W_\nu \|_M \leq \Omega |W_0 - W_\nu|_M^2,
\]

where

\[
\Omega = \left( \frac{\| M^{-1} S \| (1/4\eta)^2 + (1/4\eta)\gamma_1 \| M^{-1} L \|}{\gamma_1 + \eta \gamma_1 \| M^{-1} L \|} \right)^T \times \exp\left(2\gamma_1 + 2\eta \gamma_1 \| M^{-1} L \| \right) .
\]

where \( \gamma_1 \) and \( \gamma_2 \) are non-negative Lipschitz constants and \( k = \| W_0 \|_M \).

**Proof:** We start by subtracting the full-order models associated with the nominal and new parameters \( \nu \) and \( \nu_i \) as

\[
W_0 - W_\nu = -\nu M^{-1} SW_\nu + \nu_i M^{-1} SW_\nu + \Gamma(W_0) - \Gamma(W_\nu) + (\nu - \nu_i) M^{-1} LU(t).
\]

By multiplying both sides by \( W_0 - W_\nu \), the following is obtained:

\[
\left\{ W_0 - W_\nu, W_0 - W_\nu \right\}_M = \left\{ \left( -M^{-1} S \left( \nu W_0 - \nu_i W_0 \right) + (\nu - \nu_i) M^{-1} L U(t) \right), W_0 - W_\nu \right\}_M .
\]

This can be rewritten in the following form using \( M \) induced norm (see (30)) , where the matrix norm induced by the vector norm \( \| \cdot \|_M \) of the matrix norm is defined as \( \| B \| = \max \| Bu \|_M \| u \|_M = 1 \).

For any admissible control input \( u(t) \) and locally Lipschitz non-linearity \( f \) in \( W_\nu, W_{\nu i} \in \mathcal{D} \), (30) results in (see (31)) . Applying Young’s inequality results in

\[
\frac{1}{2} \frac{d}{dt} \| W_0 - W_\nu \|_M \leq \| M^{-1} S \| \left( -\nu \| W_0 - W_\nu \|_M^2 + \frac{1}{2} \| \nu - \nu_i \| \right) \| W_0 \|_M
\]

\[
+ \gamma_1 \| W_0 - W_\nu \|_M \ln \left( \| M^{-1} S \| \right) \| W_\nu - W_\nu \|_M + \frac{1}{2} \gamma_1 \| M^{-1} L \| \| W_\nu - W_\nu \|_M .
\]

As seen from the right hand side of the inequality above, the boundedness of each error term needs to be investigated to essentially evaluate the bound on \( \| W_0 - W_\nu \|_M \).

**Lemma 3:** The POD-Galerkin error \( \| W_0 - \Psi_{\nu i} W_{\nu i} \|\) is bounded by (see (31)) .

**Proof:** The POD-Galerkin error is broken into the Galerkin projection error \( \| W_0 - \Psi_{\nu i} W_{\nu i} \|\) and \( \| \Psi_{\nu i} W_{\nu i} - \Psi_{\nu i} W_{\nu i} \|\). The first error term was proven in [31] to be bounded on the time interval \((0, T)\) as

\[
\| W_0 - \Psi_{\nu i} W_{\nu i} \| \leq \gamma \sum_{i=1}^{N} \| W_0 - \Psi_{\nu i} W_{\nu i} \|.
\]

To investigate the boundedness of the POD-Galerkin error \( \| \Psi_{\nu i} W_{\nu i} - \Psi_{\nu i} W_{\nu i} \|\), the projection operator \( \Psi_{\nu i} \) is written in the matrix form as \( \Psi_{\nu i} = (\Psi_{\nu i})^T \) where \( \Psi_{\nu i} = (\Psi_{\nu 1}, \Psi_{\nu 2}, ..., \Psi_{\nu N}) \).

From the models (3) and (6), we obtain (see equation below).
Using the commutativity of the time derivative and the projection operator \( P_{\psi_0} \), remote space, it is concluded that
\[
\left\{ \frac{d}{dt}(W_{\psi_0} - P_{\psi_0} W_{\psi_0}), P_{\psi_0} W_{\psi_0} - \Psi_{\psi_0} \nu_{\psi_0} \right\}_M = 0.
\]
For an admissible and bounded control input \( U(t) \), the model (20) is bounded by
\[
G = \| A_{\psi_0} W_{\psi_0}(t) + b_{\psi_0}(t, W_{\psi_0}(t), U(t)) \|_W \geq 31.1
\]
Next, using the properties of the weighted inner product and applying Young's inequality considering locally Lipschitz non-linearity \( b_{\psi_0} \), we obtain the following inequality:
\[
\frac{1}{2} \frac{d}{dt} \| P_{\psi_0} W_{\psi_0} - \Psi_{\psi_0} \nu_{\psi_0} \|_M^2 \leq \left( \frac{\xi}{2} \| \Psi_{\psi_0}(\nu_{\psi_0})^T S \| \| W_{\psi_0} - P_{\psi_0} W_{\psi_0} \|_M^2 + \frac{1}{2} \| P_{\psi_0} W_{\psi_0} - \Psi_{\psi_0} \nu_{\psi_0} \|_M^2 - \nu_{\psi_0} \| \Psi_{\psi_0}(\nu_{\psi_0})^T S \| \right. \\
\times \| P_{\psi_0} W_{\psi_0} - \Psi_{\psi_0} \nu_{\psi_0} \|_M \cdot \left. + \frac{1}{2} \| P_{\psi_0} W_{\psi_0} - \Psi_{\psi_0} \nu_{\psi_0} \|_M^2 + \frac{1}{2} \| \Psi_{\psi_0}(\nu_{\psi_0})^T \| \| W_{\psi_0} - P_{\psi_0} W_{\psi_0} \|_M^2 \right) \\
+ \frac{1}{2} \| P_{\psi_0} W_{\psi_0} - \Psi_{\psi_0} \nu_{\psi_0} \|_M^2 + \frac{1}{2} \| M^{-1} - \Psi_{\nu_0}(\nu_{\psi_0})^T \| \| G \| \\
\times \| P_{\psi_0} W_{\psi_0} - \Psi_{\psi_0} \nu_{\psi_0} \|_M^2. 
\]
After rearranging (36), we obtain the following inequality:
\[
\frac{d}{dt}(W_{\psi_0} - P_{\psi_0} W_{\psi_0}) + \frac{d}{dt} \left( P_{\psi_0} W_{\psi_0} - \Psi_{\psi_0} \nu_{\psi_0} \right)
\]
\[
= \left( \nu_{\psi_0} - \Psi_{\psi_0} \nu_{\psi_0} \right) A_{\psi_0} (W_{\psi_0} - P_{\psi_0} W_{\psi_0}) \\
+ \left( M^{-1} - \Psi_{\nu_0}(\nu_{\psi_0})^T \right) A_{\psi_0} W_{\psi_0} \\
+ \left( M^{-1} - \Psi_{\nu_0}(\nu_{\psi_0})^T \right) b_{\psi_0}(t, W_{\psi_0}(t), U(t)) \\
+ \Psi_{\psi_0}(\nu_{\psi_0}) (b_{\psi_0}(t, W_{\psi_0}(t), U(t)) - b_{\psi_0}(t, W_{\psi_0}(t), U(t))).
\]
Using (35) and the Gronwall lemma, the following inequality is obtained:
\[
\frac{d}{dt} \| P_{\psi_0} W_{\psi_0} - \Psi_{\psi_0} \nu_{\psi_0} \|_M^2 \leq \left( \frac{\xi}{2} \| \Psi_{\psi_0}(\nu_{\psi_0})^T S \| \| W_{\psi_0} - P_{\psi_0} W_{\psi_0} \|_M^2 + \frac{1}{2} \| M^{-1} - \Psi_{\nu_0}(\nu_{\psi_0})^T \| \| G \| \\
\times \| P_{\psi_0} W_{\psi_0} - \Psi_{\psi_0} \nu_{\psi_0} \|_M^2 + \frac{1}{2} \| \Psi_{\psi_0}(\nu_{\psi_0})^T \| \| W_{\psi_0} - P_{\psi_0} W_{\psi_0} \|_M^2 \right) \\
+ \frac{1}{2} \| M^{-1} - \Psi_{\nu_0}(\nu_{\psi_0})^T \| \| G \| \| P_{\psi_0} W_{\psi_0} - \Psi_{\psi_0} \nu_{\psi_0} \|_M^2.
\]
To ensure the negativity of the term inside the exponential function, the following inequality should hold:
\[
\nu_{\psi_0} > \frac{(1/\xi) + 2 + 2\gamma_{\psi_0} \| \Psi_{\psi_0}(\nu_{\psi_0})^T \|}{\| \Psi_{\psi_0}(\nu_{\psi_0})^T \|}. 
\]
If inequality (37) holds true for \( \nu_{\psi_0} = \nu_\psi \), we can obtain the bound on the error as follows:
\[
\| P_{\psi_0} W_{\psi_0} - \Psi_{\psi_0} \nu_{\psi_0} \|_M^2 \leq \left( \frac{\xi}{2} \| \Psi_{\psi_0}(\nu_{\psi_0})^T S \| \| W_{\psi_0} - P_{\psi_0} W_{\psi_0} \|_M^2 + \frac{1}{2} \| M^{-1} - \Psi_{\nu_0}(\nu_{\psi_0})^T \| \| G \| \right) \| P_{\psi_0} W_{\psi_0} - \Psi_{\psi_0} \nu_{\psi_0} \|_M^2 \\
\times \sum_{i=1}^{N} \nu_{\psi_0} + \| M^{-1} - \Psi_{\nu_0}(\nu_{\psi_0})^T \| \| G \|.
\]
\[ \left\| W_{x_0} - \Psi_{v_0}^{T} W_{x_0}^{T} \right\|_{\mathcal{M}} \leq (1 + \delta \varepsilon) \| \Psi_{v_0}^{T}(\Psi_{v_0}^{T})^{T} \|_F^2 + \gamma \| \Psi_{v_0}^{T}(\Psi_{v_0}^{T})^{T} \|_F^2 )T \]
\[ \times \sum_{i=1}^{N} \lambda_{0,i} + \| M^{-1} - \Psi_{v_0}^{T}(\Psi_{v_0}^{T})^{T} \|_F^2 \]
\[ \equiv \Omega(l) \nu - \nu_0^T. \]

This concludes the proof for the boundedness of the POD-Galerkin error.\( \square \)

**Remark 2:** The bound on the last term of the error in (29), \[ \| \Psi_{v_0}^{T} W_{x_0}^{T} - \Psi_{v_0}^{T} W_{x_0}^{T} \|_{\mathcal{M}} \], is obtained as
\[ \| \Psi_{v_0}^{T} W_{x_0}^{T} - \Psi_{v_0}^{T} W_{x_0}^{T} \|_{\mathcal{M}} \leq \Omega(l) \nu - \nu_0^T. \]

where
\[ \Omega = \left( \| \Psi_{v_0}^{T}(\Psi_{v_0}^{T})^{T} \|_F^2 + \frac{1}{4\varepsilon_0} \| \Psi_{v_0}^{T}(\Psi_{v_0}^{T})^{T} L \|_F^2 \right) \times |l| \nu - \nu_0^T. \]

The proof follows the same procedure as presented in Lemma 2.

The following theorem ensures the boundedness of the model uncertainties \( \Delta \mathcal{W}_N^\nu \) and \( \Delta \mathcal{W}_N^v \).

**Theorem 1:** The error caused by the estimation of the full-order state \( W \) through a priori chosen sequence of the POD eigenvalues \( \{\lambda\}_{i=1} \) and basis functions \( \Psi_{v_0} \) at the nominal viscosity parameter \( \nu_0 \) can be estimated and bounded with respect to the variation of \( \nu \) as
\[ \left\| W_{x_0} - \Psi_{v_0}^{T} W_{x_0}^{T} \right\| \leq 2(1 + \delta \varepsilon) \| \Psi_{v_0}^{T}(\Psi_{v_0}^{T})^{T} \|_F^2 + \gamma \| \Psi_{v_0}^{T}(\Psi_{v_0}^{T})^{T} \|_F^2 )T \]
\[ \times \sum_{i=1}^{N} \lambda_{0,i} + \| M^{-1} - \Psi_{v_0}^{T}(\Psi_{v_0}^{T})^{T} \|_F^2 \]
\[ +(2\Omega + 4\delta \varepsilon) |l| \nu - \nu_0^T. \]

**Proof:** From (29), Lemmas 2 and 3, we obtain the inequality (38).\( \square \)

Inequality (38) gives an estimate of the error associated with the model reduction and varying parameter described by the POD basis functions at the nominal viscosity \( \nu_0 \). As seen from (38), the first two terms will vanish for an \( N \)-th order reduced model. In fact, the first two terms represent the error associated with the reduced-order model and the last term takes into account the effect of the changing parameter \( \nu \).

**Remark 3:** The model uncertainties \( \Delta \mathcal{W}_N^\nu \) and \( \Delta \mathcal{W}_N^v \) associated with the estimation of the POD bases \( \Psi_{v_0} \) at the nominal viscosity \( \nu_0 \) are bounded by variation of the system parameter \( \nu \) and a priori known set of system matrices as described in Theorem 1, \[ \left\| \Delta \mathcal{W}_N^\nu \right\| < \bar{B}_r \text{ and } \left\| \Delta \mathcal{W}_N^v \right\| < \bar{B}_r. \]

The error bound is calculated offline with respect to the reduced and full-order models at the nominal value \( \nu_0 \).

3.3 Proposed controller stability analysis

An SMC law is synthesised here to move the trajectories of the system onto the previously defined surface in the SMC scheme (23) in a finite time while taking into account the model uncertainties due to the changing parameter \( \nu \). The main objective can be stated as designing a reduced-order model based control law at the nominal viscosity \( \nu_0 \) with the bounded model uncertainties as obtained in the previous section. The following theorem gives an SMC that can stabilise the original full-order model.

**Theorem 2:** The proposed SMC law \( u(t) \) guarantees the asymptotic stability for the sliding surface (23) of the system represented by (20)
\[ u(t) = (CM^{-1})^T \tau(t) - C(\Psi_{v_0}^{T}(\Psi_{v_0}^{T})^{T} \nu_0) \]
\[ -\frac{1}{2} M^{-1} K(\Psi_{v_0}^{T}(\Psi_{v_0}^{T})^{T} \nu_0(t) + (\Psi_{v_0}^{T}(\Psi_{v_0}^{T})^{T} \nu_0(t) + M^{-1}F(t))) \]
\[ -(CM^{-1})^T(\lambda S + \lambda(\bar{B}_r + \bar{B}_r)\text{sign}(S)) \]

where \( \lambda > 0 \) and \( \lambda > 1 \) are the sliding mode parameters and sign(·) is the sign function.

**Proof:** The implemented Lyapunov function as below is used to prove the asymptotic stability of the proposed SMC controller (39)
\[ V(t) = \frac{1}{2} S^T S, \]

where \( S \) is the sliding surface (24). For the chosen Lyapunov function, we should have
\[ \frac{dV(t)}{dt} < 0 \text{ or } S^T S < 0, \]

in a proximity of the surface represented by \( S = 0 \). Substituting (24) and (27) into (41), we obtain
\[ S^T S = S^T (CM^{-1})^T \tau(t) + CM^{-1} F - \tau(t)) \]
\[ = S^T \left( -\lambda S + (\Delta \mathcal{W}_N^\nu + \Delta \mathcal{W}_N^v) - \lambda(\bar{B}_r + \bar{B}_r)\text{sign}(S) \right) \]

According to Remark 3, \[ \left\| \Delta \mathcal{W}_N^\nu + \Delta \mathcal{W}_N^v \right\| < (\bar{B}_r + \bar{B}_r), \]

and hence, we have
\[ S^T \left( \Delta \mathcal{W}_N^\nu + \Delta \mathcal{W}_N^v - \lambda(\bar{B}_r + \bar{B}_r)\text{sign}(S) \right) < 0, \]

where \( \lambda_i \) is a diagonal matrix with \( \lambda_1 > 1 \) and \( \lambda_2 > 1 \). This concludes the negativity of the time derivative of Lyapunov function \( V < 0 \).\( \square \)

As shown in (42), the sliding surface is directly dependent on the reduced-order model. Therefore, the reduced-order controller is capable of stabilising the full-order model considering model uncertainties arising from both model reduction and varying parameter \( \nu \). The coefficients in the proposed SMC law are tuned by trial and error gave the trade-off between reaching time and chattering. The values of these coefficients need to be large if the system trajectory is not close to the sliding surface (to ensure that the reaching time is short). In addition, they should be tuned to be small enough to reduce the chattering. It is noted that the bounds on model uncertainties associated with the estimation of the POD bases \( \Psi_{v_0} \) at the nominal viscosity \( \nu_0 \) can directly impact the performance of the closed-loop system. The obtained condition of the SMC coefficients guarantees the stability of the system, where the condition is independent of the error bounds. However, changing the error bounds as a result of varying uncertainty can impact the controlled system performance.

3.4 Functional observer design

The linear functionals of the full-order model states can be estimated by a functional observer. The reduced-order model states can be written as linear functionals of the full-order model states (FE model states), and hence, the states of the reduced-order model are estimated via a functional observer that can lead to a lower computational cost in the controller design.

The model represented in (16) is extracted by discretising the Burgers’ PDE that has locally Lipschitz non-linearities in a region represented by \( \mathcal{B} \).
i. Find the equivalent lumped-parameter model of the Burgers’ PDE as in (8) by using FE method.

ii. Develop the reduced-order model by utilising the continuous POD approach, and then describe it in the state-space form (16) in parameter value \( \nu_0 \).

iii. Design a functional observer as in (43) to estimate the states of the reduced-order model for \( \nu_0 \).

iv. Estimate the bounds on the model errors as given in (38) and Remark 3 for varying parameter \( \nu \).

v. Design the sliding mode controller based on the reduced-order model by

(a) defining the sliding surface for the reduced-order model as in (24) and finding equivalent control law \( u_{eq} \) and

(b) finding the control law \( u_{sw} + u_{eq} \) as in (39) to guarantee the closed-loop system stability.

4 Results and discussion

The numerical results of the simulated case study are illustrated in this section. The accuracy of the reduced-order model is investigated and closed-loop performance of the proposed controller is studied for two different trajectory tracking cases.

4.1 Assessment of the reduced-order model accuracy

The following example of a viscous Burgers’ equation is used for investigating the accuracy of the model. The associated forcing term in (1a) is

\[ f(t, x) = -\exp(-t)\sin(\pi x). \]

The initial condition is assumed to be

\[ w_0(x) = \begin{cases} 0.45 - 0.5\cos(8\pi x) + 0.05\cos(16\pi x), & \text{for } x \in \left(0, \frac{1}{4}\right], \\ 0, & \text{otherwise}. \end{cases} \]

Also, a set of sinusoidal boundary conditions covering frequencies up to 75 Hz are used to simulate a rich snapshot matrix that is later used to obtain the POD basis vectors. Running this kind of boundary regime excites a reasonably large number of dynamical constituents of the Burgers’ system [32]. Also, for the simulation purpose, \( N \) is chosen as 160. At first, the eigenvalue problem is solved for the nominal value of \( \nu_0 = 0.01 \) and the corresponding eigenvalues are shown in a descending order in Fig. 1. As observed, 99% of the total energy is captured by the first seven eigenvalues. Fig. 1 (right) shows the energy captured by choosing different number of eigenvalues. The numerical simulation results of the open-loop system by both FEM and 7th POD approach are illustrated for \( \nu = 0.01 \). As shown in Fig. 2, the use of POD method provides a close match with the solution of FE method. An illustrative set of basis functions representing the reduced-order model are shown in Fig. 3. In order to observe the difference between the behaviour of the models with varying viscosity, Fig. 4 shows the open-loop simulation result of the Burgers’ equation for \( \nu = 0.1 \) and the simulated response with the extracted eigenvectors at \( \nu_0 = 0.01 \). The simulated response using POD shows a different profile of the flow due to the discrepancy between true eigenvectors at \( \nu \) and extracted basis at \( \nu_0 \).
4.2 Results for the SMC design

The proposed robust sliding mode controller designed based on the reduced-order model takes the parameter variations into account. Although the reduced-order controller is designed at the nominal value of the system parameters, it can guarantee the stability of the controlled full-order system under the model and parametric uncertainties. In order to examine the performance of the proposed SMC law, the obtained full-order model is used for the validation purposes in tracking a given reference trajectory in the presence of the model uncertainties. The fluid mechanics of the underlying problem indicates that the control input is in fact, the changing flux on the boundaries to achieve the desired flow velocity at the desired points while the model uncertainties exist due to the varying viscosity and uncertainty in the reduced-order model. It is assumed that the viscosity changes from the nominal value $\nu_0 = 0.01$ to $\nu = 0.1$. The proposed controller design method computes the reduced-order model only once at $\nu_0$ reducing the computational burden. The bounds obtained on uncertainties ensure the tracking of the defined reference trajectories. The Burgers’ equation with the same initial condition as in the last section is used here. The switching and equivalent control laws are obtained from the 7th order reduced model. Control outputs in flow control problems are typically close to the boundaries, hence, the sensors are installed on points near boundaries at $y_1 = 0.895$ and $y_2 = 0.074$ to collect the flow velocity measurement. Two different functions are considered as reference inputs, a ramp and a sinusoidal function. Figs. 5 and 6a illustrate the tracking performance and the control inputs for a ramp function, respectively. As described earlier, the constant parameters of the switching control law in (39) are tuned using trial and error in such a way that a reasonable trade-off between chattering and reaching time can be achieved. After trying different combinations of the switching control parameters, the best results are obtained using the parameters $\lambda_1 = 4.3$ and $\lambda_2 = 9.3$ given the ramp reference signal. In order to examine the impact of the error bound on the tracking error, Fig. 7 illustrates the case where $\nu = 0.2$. This shows tracking error for a higher error bound due to an increased uncertainty.

Fig. 6b shows the results for the SMC design. A sinusoidal reference signal is used to assess the tracking performance. The corresponding results and the control inputs for a given signal are shown in Figs. 8 and 9a, respectively. The tuned controller parameters are $\lambda_1 = 9.6$ and $\lambda_2 = 19.8$. The result for the full-order model is shown in Fig. 9b.

As seen in Figs. 5 and 8, the proposed SMC law illustrates a good performance in terms of tracking the given signal when there are uncertainties in the model. The difference between the models arises from the fact that the eigenfunctions are obtained at $\nu_0$ for the reduced-order model. The eigenfunctions are then used to reduce the full-order model to the new viscosity of $\nu$. The switching control law is expected to keep the system on the given trajectory in the presence of the mentioned uncertainties.
5 Concluding remarks

In this work, we derived and validated a reduced-order model for the Burgers’ equation with Neumann boundary conditions, where the reduced-order model was obtained by utilising a combination of the POD-Galerkin method and weak solution approach. It was shown that extracting the POD bases associated with the reduced-order model only once at the nominal value of the viscosity parameter \( \nu_0 \) creates a source of uncertainty in the case of varying viscosity. It was proven that the error terms associated with the model reduction and changing viscosity are bounded in terms of the system matrices at the nominal value of viscosity \( \nu_0 \). The developed reduced-order model was then employed for the design of a robust sliding mode controller with respect to the defined sliding mode surfaces. Since the measurements of the states of the reduced model are required for sliding mode controller design, implementing the calculated reduced model only once at the nominal value of the viscosity significantly decreases the required computational cost for both observer and controller design. The controller design requires the computation of the reduced model for...
the varying parameter $\nu$; however, the proposed robust sliding mode controller only requires to implement the obtained reduced model at $u_0$ and guarantees the desired tracking performance of the full-order model at the new parameter $\nu$. Finally, the simulation results confirmed that the sliding mode controller led to high tracking performance.

6 References


Fig. 9 (a) SMC control inputs for tracking of a sinusoidal function, (b) Corresponding velocity response