$$-\frac{c_2 - c_3}{2} e^{(t_{k+1} - h_k - t_{l+1}) A_K(\rho(t_k))} - \frac{1 + c_3}{2} I \Big)$$
$$\times A_K^{-1}(\rho(t_k)) A_{Kh}(\rho(t_k))$$
$$A_{l+2} = \frac{c_4}{2} \left(e^{(t_{k+1} - h_k - t_{l+1}) A_K(\rho(t_k))} - I \right) A_K^{-1}(\rho(t_k)) A_{Kh}(\rho(t_k)). \tag{45}$$

As noted, the last stage of the proposed algorithm is to discretise the controller. This is inevitable since the LPV parameters are measured only at discrete instants. However, the advantage of the proposed sampled-data control is that the effect of sampling and holding devices is included in the design process. The discrete-time controller (43) is finally used in the configuration shown in Figure 1.

5.1 *Summary of the sampled-data controller design*

In order to solve the sampled-data control design problem, several intermediate steps are taken. Some of the steps are implemented offline and some in real-time. First, LMI (32) of Theorem 4.1 is solved. By solving this LMI problem, the basis functions associated with LMI variables, namely $X, Y, \hat{A}, \hat{A}_h, \hat{A}_\tau, \hat{B}, \hat{C}$ and D_K (which could be parameter-dependent) are determined for second stage. At each sampling instant, the scheduling parameter is measured and the aforementioned LMI variables are updated accordingly. Employing steps 1 and 2 in Theorem 4.1, the continuous-time controller matrices, i.e., A_K, A_{Kh}, $A_{K\tau}$, B_K, C_K and D_K are determined. For implementation purpose, the digital controller (43) is to be found. This is carried out by means of Equation (44) and the most recent samples of the controller states within the previous h_m seconds.

6. Illustrative example

As an illustrative example, we design a sampled-data controller to control the chattering of a milling machine, whose simplified mechanical model is depicted in Figure 3. The dynamic model of this system can be formulated as an LPV system containing a parameter-dependent time delay (Tan, Grigoriadis, & Wu, 2003; Zhang et al., 2002; Zope et al., 2012).

The system consists of a two-blade cutter of mass m_1 and a spindle of mass m_2. Also two springs with stiffness k_1 and k_2 and a damping with the coefficient c are lumped in the model. The rotation of the cutter causes the removal of workpiece material from the surface resulting in a force acting on the cutter denoted by f in the figure. If no control force is applied to the spindle, the machine exhibits the chattering. To reduce the chattering during the milling process, a force u is to be applied to the spindle dictated by a controller. To this aim, we first derive the dynamic equation associated with the model in Figure 3. Introducing the

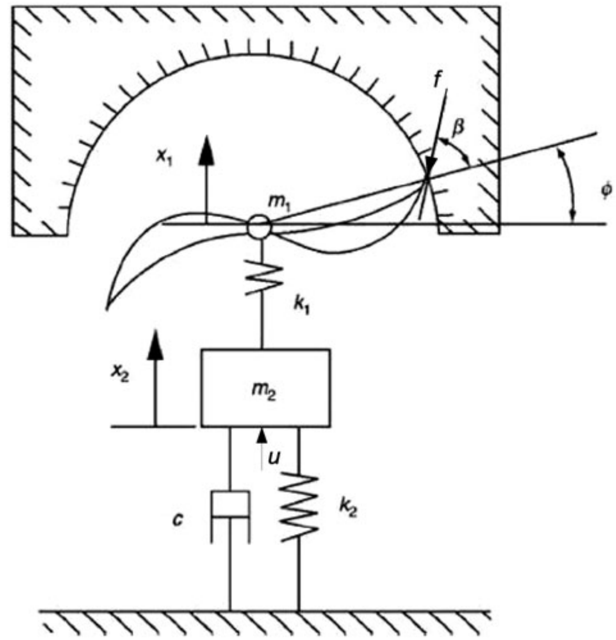

Figure 3. A simplified schematic of milling process.

displacement and velocity of the cutter and spindle as state variables, the dynamic model of the system is described by

$$\begin{aligned} m_1\ddot{x}_1 + k_1(x_1 - x_2) &= f \sin(\phi + \beta) + w \\ m_2\ddot{x}_2 + c\dot{x}_2 + k_1(x_2 - x_1) + k_2 x_2 &= u, \end{aligned} \tag{46}$$

where w represents the external disturbance input. Modelling the surface at the point, where the blade touches the surface with a spring of stiffness k, the displacement of this spring equals the difference between the tip position of the now touching blade and that of the previous blade at that point. Assuming that the angular velocity of the cutter $\omega(t)$ remains constant in a rotation, the passage time between two blades is equal to $\frac{\pi}{\omega}$. As a result, the corresponding reaction force of the surface is

$$f = -k\left[x_1(t) - x_1\left(t - \frac{\pi}{\omega}\right)\right] \sin(\phi).$$

Therefore, we can rewrite the system equations in Equation (46) as

$$\begin{aligned} \ddot{x}_1 &= \frac{1}{m_1}[-k_1 - k\sin(\phi)\sin(\phi+\beta)]x_1 \\ &\quad + \frac{k_1}{m_1}x_1\left(t - \frac{\pi}{\omega}\right) + \frac{k_1}{m_1}x_2 + \frac{k_1}{m_1}w(t) \\ \ddot{x}_2 &= \frac{k_1}{m_2}x_1 - \frac{k_1 + k_2}{m_2}x_2 - \frac{c}{m_2}\dot{x}_2 + \frac{k_1}{m_2}u. \end{aligned} \tag{47}$$

Table 1 summarises the data corresponding to this example.

Table 1. The milling system parameters.

Parameter	Value	Unit
m_1	1	Kg
m_2	2	Kg
k_1	10	N/m
k_2	20	N/m
k	3	N/m
c	0.5	N/m s
β	70	Degree

The obtained model relies on two measurable parameters ϕ and ω. First, we note that

$$\begin{aligned}\sin(\phi)\sin(\phi+\beta) &= 0.5[\cos(\beta)-\cos(2\phi+\beta)]\\ &= 0.1710 - 0.5\cos(2\phi+\beta).\end{aligned}$$

Next, we define the scheduling parameter vector as $\rho(t) = [\rho_1(t), \rho_2(t)]^T$ with $\rho_1(t) = \cos(2\phi(t) + \beta)$ and $\rho_2(t) = \omega(t)$. The rotation speed ω is assumed to vary between 200 rpm (20.94 rad/sec) and 1000 rpm (104.7 rad/sec). The parameter space associated with the LPV parameters is as $\rho_1(t) \in [-1\ 1]$ and $\rho_2(t) \in [20.94\ 104.7]$. For the parameter-dependent time delay $h(t) = \pi/\omega(t)$, we have

$$0.015 < h(t) < 0.15,\ |\dot{h}(t)| = \left| -\frac{\pi}{\omega^2} \times \dot{\omega} \right| \le 0.75.$$

Considering the state vector to be $x = [x_1\ x_2\ \dot{x}_1\ \dot{x}_2]^T$, the state-space LPV representation of the system is

$$\begin{aligned}\dot{x}(t) &= \begin{bmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -10.34+\rho_1(t) & 10 & 0 & 0\\ 5 & -15 & 0 & -0.25\end{bmatrix} x(t)\\ &\quad + \begin{bmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0.34-\rho_1(t) & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} x\left(t - \frac{\pi}{\rho_2(t)}\right)\\ &\quad + \begin{bmatrix}0\\0\\1\\0\end{bmatrix} w(t) + \begin{bmatrix}0\\0\\0\\0.5\end{bmatrix} u(t)\\ z(t) &= \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} x(t) + \begin{bmatrix}0\\0\\0.1\end{bmatrix} u(t)\\ y(t) &= \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{bmatrix} x(t), \end{aligned} \qquad (48)$$

where $z(t)$ is a fictitious system output reflecting the control design objectives. We seek for an $\mathcal{H}_\infty$ controller to reduce the displacement of the elements, as well as penalise large control actions.The measurement vector $y(t)$ includes only the positions of the spindle and cutter. For effective control of the system, the choice of sampling frequency is crucial and is a trade-off between the quality of the closed-loop system response and the implementation costs. Since two blades touch the surface twice per revolution, it is quite reasonable that the sampling frequency be at least twice in a rotation. It is noted that since the angular velocity might vary during the milling process, the sampling period is not fixed and changes accordingly. In this example, we consider two samples per revolution, i.e.,

$$t_{k+1} = t_k + \frac{1}{2}\frac{2\pi}{\omega(t_k)}, \qquad (49)$$

with $t_0 = 0$. Considering the bounds on the time delay in Equation (48), we have $h_m = 0.15$ and $\tau_m = 0.15$ sec to be used in the LMI (32). In addition, we have to decide on the structure of the function variables involved in this LMI problem. The structure of the matrix functions

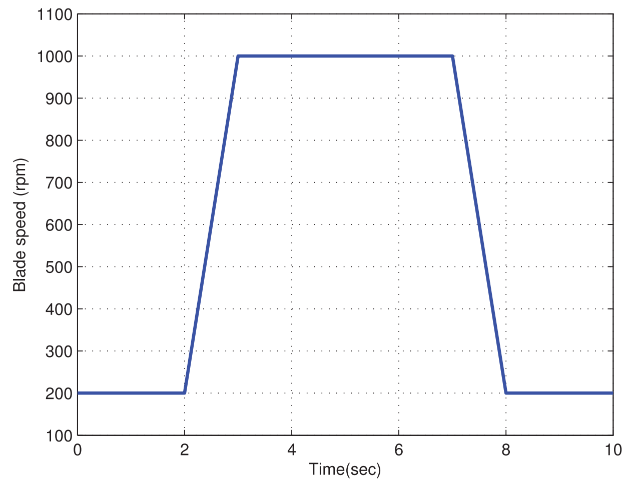

Figure 4. Blade rotation speed profile (rpm).

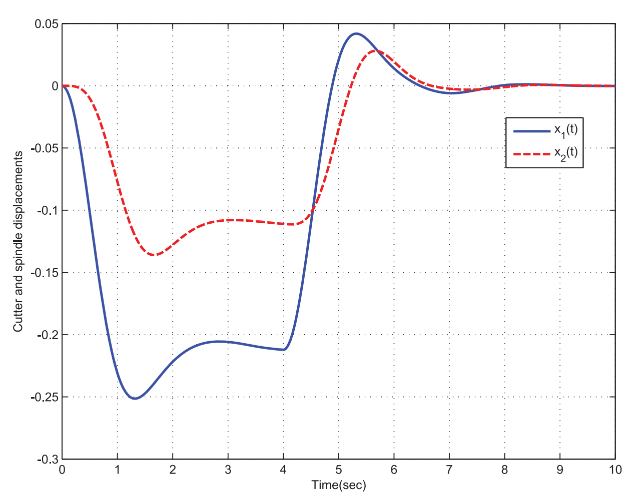

Figure 5. Displacement of cutter and spindle.

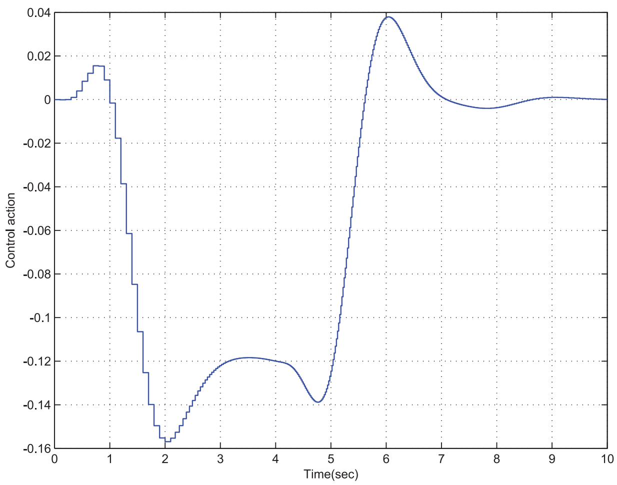

Figure 6. Control effort.

is part of the design and is an ad hoc procedure. One obvious choice is to consider constant function variables (parameter-independent) to reduce the computational cost. However, here the structure of LMI variables is chosen according to the direction given in Remark 3. It is noted that the optimal value of γ is sensitive to the value of the scalars λ_2, λ_3 and λ_4. These scalars can be optimised by performing a three-dimensional search. For the milling example, our search resulted in $\lambda_2 = 10$, $\lambda_3 = 1$ and $\lambda_4 = 1$. Also the obtained $\mathcal{H}_\infty$ performance level was calculated to be $\gamma = 0.56$.

It is assumed that the system is perturbed by a rectangular disturbance $w(t)$ of magnitude one over the time interval $t \in [0, 4]$ and zero elsewhere. The blade rotational speed profile is shown in Figure 4. Shown in Figure 5 is the simulation result of the proposed control scheme for the milling machine example, indicating the displacement of the cutter x_1 and that of the spindle x_2 for a predefined test condition. It is apparent that the proposed controller attenuates the disturbance successfully under the variable rotational speed. The control effort required for this study is shown in Figure 6, in which the different lengths of sampling periods is a result of using Equation (49). Finally, for comparison, we examine the open-loop response of the milling machine, while no control commands exist. It is assumed that the system is perturbed by the same disturbance signal as in the previous simulation. Figure 7 shows the displacement of the masses that demonstrate the long time it takes for them to vanish. This justifies the use of a controller to eliminate the fluctuations and reduce the settling time.

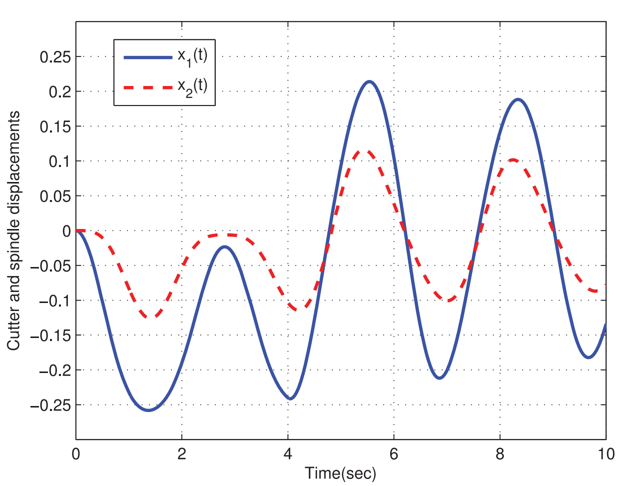

Figure 7. Open-loop response of the milling machine.

7. Conclusion and future work

In this paper, we employ the input-delay approach for sampled-data control design of state-delayed continuous-time LPV systems by mapping the hybrid closed-loop system into a continuous-time state-delay LPV system. It is shown that appropriate choice of a Lyapunov–Krasovskii functional concludes a delay-dependent synthesis condition that can handle the fast-varying time delay successfully. To ensure the asymptotic stability and $\mathcal{H}_\infty$ performance of the resulting closed-loop delay system, we utilise slack variables to relax the resulting inequality condition in the form of an LMI problem. We further show that the proposed sampled-data control design method can handle the output-feedback control design even for systems with varying sampling rates. Finally, by means of a numerical example, we demonstrate that the proposed method can handle the sampled-data control problem even with varying sampling periods. The authors are currently investigating to further reduce the conservatism of the results, e.g., by using the Wirtinger's inequality in the Lyapunov–Krasovskii functional.

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