A dynamic feedback control strategy for control loops with time-varying delay

Behrouz Ebrahimia, Reza Tafreshi a,∗, Matthew Franchek b, Karolos Grigoriadis b and Javad Mohammadpour c

aDepartment of Mechanical Engineering, Texas A&M University at Qatar, Doha 23874, Qatar; bDepartment of Mechanical Engineering, University of Houston, Houston, TX 77204, USA; cCollege of Engineering, University of Georgia, Athens, GA 30602, USA

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Dynamic systems of nth order with time-varying delay in the control loop are examined in this paper. The infinite-dimensional pure delay problem is approximated using a jth-order Padé approximation. Although the approximation provides a well-matched finite-dimensional configuration, it poses a new challenge in terms of unstable internal dynamics for the resulted non-minimum phase system. Such a non-minimum phase characteristic limits the closed-loop system bandwidth and leads to an imperfect tracking performance. To circumvent this problem, the unstable internal dynamics of the system is captured and a new dynamic compensator is proposed to stabilise it in a systematic framework. A dynamic controller is developed, which provides the overall system stability against unmatched perturbation and meets the desired tracking error dynamics. The proposed approach is then applied to fuelling control in gasoline engines addressing the varying transport delay of the oxygen-sensor measurement in the exhaust. The developed methodology is finally validated on a Ford F-150 SI lean-burn engine model with large time-varying delay in the control loop.

Keywords: time-varying delay; feedback control; dynamic compensator; non-minimum phase systems

1. Introduction

Control systems usually operate in the presence of time delay, which is the time needed to acquire the relevant information for making and executing control decisions (Sipahi, Niculescu, & Abdallah, 2011). A feedback system, which is stable without delay may experience significant performance deterioration or even becomes destabilised while subject to some delay (Niculescu & Gu, 2004). Some examples of time-delay systems extensively studied include internal combustion engines (Ebrahimi et al., 2012), electrical motors (Prashanth & Khorrami, 2008) and inverters (Cortes, Rodriguez, Silva, & Flores, 2012), robotics (Chang & Lee, 1996), tele-surgery (Speich & Rosen, 2004), unmanned vehicles (Munz, Papachristodoulou, & Allgower, 2009), communication systems (Lehmann & Lunze, 2012), decentralised and collaborative control of multiagents (Ren & Beard, 2004), haptics (Cheong, Niculescu, Annaswamy, & Srinivasan, 2007), adaptive combustion control (Yildiz, Annaswamy, Yanakiev, & Kolmanovskii, 2010), and chemical processes with transport delays (Bozorg & Davison, 2006).

Stability analysis and control design for time-delay systems has been the subject of many important practical and theoretical problems for decades (Dugard & Verriest, 1998; Gu, Kharitonov, & Chen, 2003; Kolmanovskii & Myshkis, 1999; Tafreshi et al., 2013). Realisation of delay effects and designing stabilising controllers have been divided into two main directions; namely, delay-independent stability and delay-dependent stability criteria. However, the effect of delays becomes more pronounced for delay-dependent analyses, which are typically less conservative than the delay-independent ones. In the past decade, there have been considerable efforts devoted to the stability analysis of systems with delay (Loiseau, Niculescu, Michiels, & Sipahi, 2009; Mohammadpour & Grigoriadis, 2008; Sakthivel, Mathiyalagan, & Anthoni, 2012). Lyapunov-Krasovskii functionals (Kharitonov, 2004) and Lyapunov-Razumikhin functions (Jankovic, 2001) have been frequently used for stability analysis of time-delay systems.

In this paper, we use the jth-order Padé approximation to transfer the actual infinite-dimensional time-delay system into a finite-dimensional form, where the delay is considered as a time-varying parameter. A new dynamic compensator is then proposed to stabilise the unstable internal dynamics. The presented compensator enables the closed-loop system to track the desired tracking error dynamics while making it robust against unmatched perturbations. The control input constructed by the dynamic compensator provides the actual system with the robustness and zero steady-state tracking error properties. The contributions of the present paper are as follows: (1) the design of a parameter-varying control scheme for nth-order systems with variable input delay based on a parameter-varying dynamic compensation, which is robust against unmatched perturbations;
(2) development of a general design approach for the parameter-varying dynamic compensator to stabilise the system’s internal dynamics; and (3) examination of the applicability of the proposed control to both delayed and non-minimum phase systems with no need to tune the control gains.

The outline of the paper is as follows. Section 2 presents the system dynamics reconfigured into a form appropriate for the compensator and corresponding controller design. Application of the proposed method to the fuelling control problem for spark ignition engines with time-varying delay is described in Section 3. Finally, Section 4 concludes the paper.

2. The control design

Consider an nth-order single-input single-output (SISO) dynamic system with a time-varying delay in the control loop as

\[
\frac{d^n}{dt^n} y(t) + \sum_{i=0}^{n-1} \alpha_i \frac{d^i}{dt^i} y(t) = \beta u(t - \tau),
\]

where \( \tau > 0 \) is a time-varying delay and \( y(t), u(t) \in \mathcal{R} \) are the system output and input, respectively. The solution of Equation (1) is infinite dimensional due to the delay inclusion in the control loop. A possible way to solve the infinite-dimensional problem of the pure delay is the use of Padé approximation. This approximation represents the system (1) in a finite-dimensional form at the expense of introducing unstable internal dynamics into the system dynamics. Such a system, which is generally referred to as a non-minimum phase system, restricts the application of classical control design techniques and remains as a challenging task in the control area. In the present paper, we use a jth-order Padé approximation as

\[
\sum_{i=0}^{j} \beta_i(\tau) \frac{d^i}{dt^i} u(t - \tau) \simeq \sum_{i=0}^{j} (-1)^i \beta_i(\tau) \frac{d^i}{dt^i} u(t),
\]

where \( \beta_i(\tau) = \frac{(2 j - i)!}{(j - i)! i!} \tau^i \).

Substituting Equation (2) into Equation (1) results in

\[
\frac{d^{n+j}}{dt^{n+j}} \hat{y}(t) + \sum_{i=0}^{n+j-1} \hat{\alpha}_i(\tau) \frac{d^i}{dt^i} \hat{y}(t) = \sum_{i=0}^{j} (-1)^i \hat{\beta}_i(\tau) \frac{d^i}{dt^i} u(t),
\]

where \( \hat{\alpha}_i/\hat{\beta}_i \) s are the delay-dependent coefficients, \( \hat{\beta}_i(\tau) = \beta \beta_i(\tau)/\beta_j(\tau) \) and \( \hat{\gamma}(t) \) is the approximated output. Equation (3) is a non-minimum phase system whose order has been increased to \( n + j \). Equation (3) can be represented in the state-space form as

\[
\dot{x}_1(t) = x_2(t),
\]

\[
\vdots
\]

\[
\dot{x}_{n+j}(t) = \sum_{i=0}^{n+j-1} \hat{\alpha}_i(\tau) x_{i+1}(t) + u(t),
\]

\[
\hat{y}(t) = [\hat{\beta}_0(\tau), -\hat{\beta}_1(\tau), \ldots, (-1)^{j-1} \hat{\beta}_{j-1}(\tau), 0, \ldots, 0] x(t).
\]

Obviously, the relative degree of Equation (4) is now smaller than the system order \( n + j \). For the state-space representation (4), the relative degree \( r \) can be obtained using Lie notation (Isidori, 1995), if

\[
L_g L_f^k h(x) = 0, \quad 0 \leq k < r - 1,
\]

\[
L_g L_f^{r-1} h(x) \neq 0,
\]

where \( f(x) = [x_2(t), \ldots, \sum_{i=0}^{n+j-1} \hat{\alpha}_i(\tau) x_{i+1}(t)]^T, g(x) = [0, \ldots, 0, 1]^T, h(x) = \hat{y}(t) \).

For the system (4) with order of \( n + j \), the relative degree is \( r = n \). Hence, there is a jth-order internal dynamics whose stability influences the stability of the system (4). The following proposition represents the system in terms of nth-order input/output pairs and jth-order internal dynamics.

**Proposition 1:** Consider the set

\[
\phi_1(x) = h(x),
\]

\[
\phi_2(x) = L_f h(x),
\]

\[
\vdots
\]

\[
\phi_n(x) = L_f^{n-1} h(x).
\]

A set of \( j \) more functions \( \phi_{n+1}(x), \ldots, \phi_{n+j}(x) \) exists such that the mapping \( \Phi(x) = [\phi_1(x), \ldots, \phi_{n+j}(x)]^T \) has non-singular Jacobian matrix. Furthermore, it is always possible to find \( \phi_{n+1}(x), \ldots, \phi_{n+j}(x) \) such that \( L_g \phi_{n+i}(x) = 0, 1 \leq i \leq j \).

**Proof:** Since the one-dimensional distribution \( G = \text{span}(g) \) is non-singular and involutive, according to Frobenius Theorem (Isidori, 1995), one can obtain \( n + j - 1 \) real-valued functions, \( \phi_1(x), \ldots, \phi_{n+j-1}(x) \), such that

\[
\text{span}\{d\phi_1, \ldots, d\phi_{n+j-1}\} = G^1,
\]

where \( G^1 \) is the annihilator of the distribution \( G \) and \( d = \frac{d}{dx} \). Suppose \( G \cap (\text{span}(dh, dL_f h, \ldots, dL_f^{n-1} h)) \neq \{0\} \) which means that the vector \( g \) annihilates all the coevals in \( \text{span}(dh, dL_f h, \ldots, dL_f^{n-1} h) \). This contradicts the fact that \( L_g L_f^{n-1} h(x) \) is non zero according to Equation (5b).
Hence,
\[
\dim(\mathbb{G}^2 + \text{span}\{dh, dL_f h, \ldots, dL_f^{n-1} h\}) = n + j. \quad (8)
\]

By using Equations (7) and (8) and the fact that the row vectors \( dh, dL_f h, \ldots, dL_f^{n-1} h \) are linearly independent with span's dimension of \( n \), the \( j \) functions \( \phi_1, \ldots, \phi_j \) in the set \( \{\phi_1, \ldots, \phi_{n+j-1}\} \) can be chosen so that the \( n + j \) differentials \( dh, dL_f h, \ldots, dL_f^{n-1} h, d\phi_1, \ldots, d\phi_j \) are linearly independent, and
\[
\{d\phi_{n+i}(x), g(x)\} = L_g \phi_{n+i}(x) = 0, 1 \leq i \leq j. \quad (9)
\]

This completes the proof.

Now, consider the system in the new coordinates \( z_i = \phi_i(x), 1 \leq i \leq n + j \). The new system description for \( 1 \leq i \leq n \) can be expressed as
\[
\begin{align*}
\dot{z}_1(t) &= L_f h(\Phi) = \phi_2(\Phi) = z_2(t), \\
\vdots \\
\dot{z}_{n-1}(t) &= L_f^{n-1} h(\Phi) = \phi_n(\Phi) = z_n(t), \\
\dot{z}_n(t) &= L_f h(\Phi) + L_g L_f^{n-1} h(\Phi) u(t),
\end{align*}
\]
where \( \Phi = \Phi^{-1}(z(t)) \). Moreover, the description of the system in the new coordinates for \( n + 1 \leq i \leq n + j \) can be written as
\[
\dot{z}_i(t) = L_f \phi_i(\Phi). \quad (11)
\]
Hence, the system dynamics (4) can be expressed in the new coordinates,
\[
z_i(t) = x_{i+j}(t) + \sum_{k=0}^{i-1} \beta_k(\tau)x_{i+k}(t), i = 1, \ldots, n. \quad (12)
\]

For the other \( j \) new coordinates, we choose \( z_{n+i} = x_i, i = 1, \ldots, j \), which indeed satisfies Proposition 1. The overall system dynamics can be then expressed as
\[
\begin{align*}
\dot{\xi}(t) &= R(\tau)\xi(t) + S(\tau)\eta(t) + \beta(\tau)u(t), \\
\dot{\eta}(t) &= P(\tau)\xi(t) + Q(\tau)\eta(t), \\
\dot{y}(t) &= [1, 0, \ldots, 0, \xi(t)],
\end{align*}
\]
where \( R(\tau) \in \mathbb{R}^{n \times n}, S(\tau) \in \mathbb{R}^{n \times 1}, P(\tau) \in \mathbb{R}^{j \times n}, \) and \( Q(\tau) \in \mathbb{R}^{j \times j} \) are delay-dependent matrices, \( \xi = [z_1 \ldots z_n]^T \), \( \eta = [z_{n+1} \ldots z_{n+j}]^T \), and \( \dot{y}(t) \) is the approximated output from Equation (3). The first equation of Equation (13) is generally referred to as input/output pairing and explicitly relates the input to the output of the system. The second \( j \)-th order equation of Equation (13) is called the internal dynamics of the system (4) and is implicitly influenced by the first \( n \)-th order equation of Equation (13). To check the stability of the internal dynamics, the zero dynamics of the system may be determined by zeroing the output of the system \( (\xi(t) = 0) \) and its successive derivatives in the second equation of Equation (13), i.e., \( \dot{\eta}(t) = Q(\tau)\eta(t) \), where
\[
Q(\tau) = \begin{bmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & 0 & \ldots \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1 \\
-\hat{\beta}_0(\tau) & \hat{\beta}_1(\tau) & -\hat{\beta}_2(\tau) & \ldots & (-1)^j \hat{\beta}_{j-1}(\tau)
\end{bmatrix}
\]

(14)

The non-Hurwitz matrix \( Q(\tau) \) has positive real part eigenvalues with the characteristic polynomial \( p(s) = -\hat{\beta}_0(\tau) - \hat{\beta}_1(\tau)s + \ldots + (-1)^j \hat{\beta}_{j-1}(\tau)s^{j-1} + (-1)^j \hat{\beta}_j(\tau)s^j \). Hence, the unstable zero dynamics, \( \dot{\eta}(t) = Q(\tau)\eta(t) \), characterises Equation (13) as a non-minimum phase system. The control design task for a non-minimum phase system is still a challenging problem because of the unstable internal dynamics. The perfect tracking for such systems is impossible since in addition to meeting the desired performance, the control system should also stabilise the internal dynamics. Such excessive control responsibility prevents application of many conventional control techniques, which have been primarily developed for minimum phase systems, where the controller operates on the tracking error signal (Shkolnikov & Shtessel, 2001). In this paper, we design a feedback control system which operates on the compensated error signal instead.

The compensated tracking error signal in fact stabilises the internal dynamics. Such excessive control responsibility prevents application of many conventional control techniques, which have been primarily developed for minimum phase systems, where the controller operates on the tracking error signal (Shkolnikov & Shtessel, 2001). In this paper, we design a feedback control system which operates on the compensated error signal instead.

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\[ \frac{d^m \kappa_i(t)}{dt^m} = 0. \] The same assumption is made for the reference input profile \( y^*(t) \).

To investigate the control design process and to stabilise the unstable internal dynamics (15), a dynamic compensator is first proposed in the following theorem:

**Theorem 1:** Consider the following dynamic compensator which operates on the tracking error \( e(t) \) and the input to the internal dynamics (15),

\[
\left( \frac{d^{k+j}}{dt^{k+j}} + \sum_{i=1}^{j} \gamma_{k+j-i}(t) \frac{d^{k+j-i}}{dt^{k+j-i}} \right) u_n(t) = \sum_{i=0}^{k-1} \lambda_i(t) \frac{d^i}{dt^i} e(t),
\]

where \( k = m - j \) and \( m \) is the order of the desired tracking error dynamics,

\[
\left( \frac{d^m}{dt^m} + \sum_{i=0}^{m-1} c_i(t) \frac{d^i}{dt^i} \right) e(t) = 0. \tag{17}
\]

We note that \( m \geq 1 \) and \( c_i \)'s are chosen based on the desired delay-dependent eigenvalue placement, and

\[
\begin{align*}
\lambda_0(t) &= \hat{\beta}_0^{-1}(t)c_0(t), \\
\lambda_1(t) &= \hat{\beta}_0^{-1}(t)c_1(t) - \lambda_0 \hat{\beta}_1(t)), \\
& \vdots \\
\lambda_i(t) &= \hat{\beta}_0^{-1}(t)c_i(t) - \lambda_{i-1}(t) \hat{\beta}_1(t) - \lambda_{i-2}(t) \hat{\beta}_2(t) - \cdots - \lambda_{i-j}(t) \hat{\beta}_j(t), \\
& \vdots \\
\lambda_{k-1}(t) &= \hat{\beta}_0^{-1}(t)c_{k-1}(t) - \lambda_{k-2}(t) \hat{\beta}_1(t) - \lambda_{k-3}(t) \hat{\beta}_2(t) - \cdots - \lambda_{k-j}(t) \hat{\beta}_j(t).
\end{align*}
\]

and

\[
\begin{align*}
\gamma_{k+j-i}(t) &= c_{k+j-i}(t) - \lambda_{k-1}(t) \hat{\beta}_{j-i+1}(t) \\
& - \lambda_{k-2}(t) \hat{\beta}_{j-i+2}(t) - \cdots - \lambda_{k-i}(t) \hat{\beta}_j(t), \\
& \vdots \\
\gamma_{k+j-1}(t) &= c_{k+j-1}(t) - \lambda_{k-1}(t) \hat{\beta}_j(t).
\end{align*}
\]

The dynamic compensator (16) stabilises the unstable internal dynamics in Equation (15) while tracking the desired reference profile \( e(t) \) and is robust against the unmatched perturbation \( \kappa(t) \).

**Proof:** Consider the system internal dynamics (15) rewritten in the following form:

\[
\sum_{i=0}^{j} (-1)^i \hat{\beta}_i(t) \frac{d^i}{dt^i} u_n(t) \\
= -e(t) + y^*(t) - \sum_{i=1}^{j} (-1)^i \hat{\beta}_i(t) \kappa_i(t), \tag{20}
\]

where \( e(t) = y^*(t) - \hat{y}(t) \). By substituting Equation (16) into Equation (20), we obtain

\[
\left[ \sum_{i=0}^{j} \sum_{p=0}^{k} (-1)^p \hat{\beta}_i(t) \lambda_p(t) \frac{d^{p+i}}{dt^{p+i}} \\
+ \sum_{i=1}^{j} \gamma_{k+j-i}(t) \frac{d^{k+j-i}}{dt^{k+j-i}} + \frac{d^{k+j}}{dt^{k+j}} \right] e(t) \\
= \left( \sum_{i=1}^{j} \gamma_{k+j-i}(t) \frac{d^{k+j-i}}{dt^{k+j-i}} + \frac{d^{k+j}}{dt^{k+j}} \right) \Pi(t). \tag{21}
\]

where \( \Pi(t) = y^*(t) - \sum_{i=1}^{j} (-1)^i \hat{\beta}_i(t) \kappa_i(t) \). Based on Assumption 1, the term in the right-hand side of Equation (21) is cancelled out and, hence, robustness of the compensator is guaranteed. Now, the dynamics of the tracking error \( e(t) \) is obtained as

\[
\left[ \sum_{i=0}^{j} \sum_{p=0}^{k} (-1)^p \hat{\beta}_i(t) \lambda_p(t) \frac{d^{p+i}}{dt^{p+i}} \\
+ \sum_{i=1}^{j} \gamma_{k+j-i}(t) \frac{d^{k+j-i}}{dt^{k+j-i}} + \frac{d^{k+j}}{dt^{k+j}} \right] e(t) = 0. \tag{22}
\]

Rearranging Equation (22) in descending order of derivatives and replacing the corresponding coefficients from Equations (18) and (19) lead to the desired error dynamics (17). \( \square \)

The presented dynamic compensator (16) will be used to design the controller that is able to track the desired reference profile while stabilising the internal dynamics of the non-minimum phase system (13).

**Theorem 2:** Consider the following dynamic control law,

\[
\left( \frac{d^{k+j}}{dt^{k+j}} + \sum_{i=0}^{j} \gamma_{k+i}(t) \frac{d^{k+i}}{dt^{k+i}} \right) u(t) \\
= \sum_{i=0}^{k-1} \lambda_i(t) \hat{\alpha}_i(t) \frac{d^{p+i}}{dt^{p+i}} e(t). \tag{23}
\]

The dynamic control (23) built upon the dynamic compensator (16) guarantees the stability of the system (13).
3. Case study: lean-burn engines with large time-varying delay

Lean-burn spark ignition engines exhibit significant performance enhancement in terms of tailpipe emissions and fuel economy compared to the traditional spark ignition engines (Zhang, Grigoriadis, Franchek, & Makki, 2007). They operate at up-stoichiometric air–fuel ratio (AFR) leading to reduced carbon monoxide and hydrocarbons but increased nitrogen oxide (NOx) levels. The excessive NOx is stored in the lean NOx trap (LNT) module which is integrated with the three-way catalyst downstream the universal exhaust gas oxygen (UEGO) sensor. The stored NOx is released after reaching a certain threshold, while simultaneous switching of the engine into rich operation converts it to non-polluting nitrogen. Although this process leads to a significant reduction in harmful emissions, it introduces a larger time-varying delay for the gas exiting the cylinder to reach the UEGO sensor. The large time delay, however, restricts the closed-loop system’s stability and bandwidth. Moreover, the wide range of engine operating conditions, the inherent nonlinearities of the combustion process, the large modeling uncertainties, and parameter variations pose further challenges to the design of the control system for lean-burn engines.

Figure 1 represents the system closed-loop configuration with engine AFR dynamics as (Ebrahimi et al., 2012; Zhang et al., 2007)

\[ \tau_s \dot{y}(t) + y(t) = u(t - \tau) \]

where \( y(t) \) and \( u(t) \) are the measured and input AFR, respectively. The parameter \( \tau_s \) is the time constant of the UEGO sensor and \( \tau \) is the overall time delay consisting of: (1) cycle delay, \( \tau_c \), which is estimated by one engine cycle due to the four strokes of the engine as \( \frac{720}{3600/60N} = \frac{120}{N} \) (sec), where \( N \) is the engine speed in revolutions per minute (rpm) and (2) gas transport delay, which is identified as the time it takes for the exhaust gas to reach the tailpipe UEGO sensor downstream the LNT, and can be approximated by \( \tau_g = \theta / \dot{m_a} \) for an average exhaust temperature, where \( \dot{m_a} \) is the air mass flow, and \( \theta \) is a constant that can be determined based on the experimental data (Zhang et al., 2007). Equation (24) matches Equation (1) with \( n = 1 \). Without loss of generality, we consider the first-order \( (j = 1) \) and second-order \( (j = 2) \) Padé approximations to design the corresponding controllers for the output tracking problem.

3.1 First-order Padé approximation

By using the first-order Padé approximation, i.e., \( j = 1 \), Equation (4) can be rewritten as

\[
\begin{align*}
\dot{x}_{1j1}(t) &= x_{2j1}(t), \\
\dot{x}_{2j1}(t) &= -a_{0j1}(\tau)x_{1j1}(t) - a_{1j1}(\tau)x_{2j1}(t) + u_{j1}(t), \\
\dot{y}_{j1}(t) &= \hat{b}_{0j1}(\tau)x_{1j1}(t) + \hat{b}_{1j1}(\tau)x_{2j1}(t),
\end{align*}
\]

(25)

where \( y_{j1}(t) \) is the approximated output with the first-order Padé approximation and \( \tilde{a}_{0j1}(\tau) = \hat{b}_{0j1}(\tau) = 2(\tau, \tau)^{-1}, \tilde{a}_{1j1}(\tau) = (2\tau + \tau)(\tau, \tau)^{-1}, \) and \( \hat{b}_{1j1}(\tau) = -\tau^{-1} \) are the delay-dependent coefficients. By using definition of Equation (5), one can easily determine that the relative degree of the second-order system (25) is \( r = 1 \). Equation (13) is used to obtain the system’s internal dynamics and the input/output pairs by choosing \([z_{1j1}(t), z_{2j1}(t)]^T = [\hat{b}_{0j1}(\tau)x_{1j1}(t) + \hat{b}_{1j1}(\tau)x_{2j1}(t), x_{1j1}(t)]^T \). Hence, Equation (24) can be expressed as

\[
\begin{bmatrix}
\dot{\xi}_{j1}(t) \\
\dot{\eta}_{j1}(t)
\end{bmatrix}
= \begin{bmatrix}
R_{j1}(\tau) & S_{j1}(\tau) \\
P_{j1}(\tau) & Q_{j1}(\tau)
\end{bmatrix}
\begin{bmatrix}
\xi_{j1}(t) \\
\eta_{j1}(t)
\end{bmatrix}
+ \begin{bmatrix}
\beta_{j1}(\tau) \\
0
\end{bmatrix} u_{j1}(t),
\]

(26)

where \( \xi(t) \) is the system output and \( \eta(t) \) is the system internal dynamics. The corresponding parameter-varying coefficients are \( R_{j1}(\tau) = -(4\tau_s + \tau)(\tau, \tau)^{-1}, S_{j1}(\tau) = (8\tau_s + 4\tau)(\tau, \tau)^{-2}, P_{j1}(\tau) = -\tau_s, \) and \( Q_{j1}(\tau) = 2\tau^{-1}. \) The eigenvalue of zero dynamics is \( \sigma = 2\tau^{-1} \), which in fact is obtained directly by zeroing the system output within Equation (26). For all positive time delay, the eigenvalue of the system is \( \sigma > 0 \), which demonstrates instability of internal dynamics. We use the integral of time multiplied by absolute tracking error (ITAE: \( J = \int_0^T t |e(t)| dt \)) criterion to determine the desired error dynamics as \( \dot{e}(t) + c_{1j1}(\tau)\dot{e}(t) + c_{0j1}(\tau)e(t) = 0 \), where \( c_{1j1}(\tau) = 1.4\omega_{a1}(\tau), c_{0j1}(\tau) = \omega_{a1}^2(\tau), \) and \( \omega_{a1}(\tau) \) is the delay-dependent natural frequency that should be determined. By using Assumption 1 with \( k = l = 1 \), the proposed dynamic compensator in Theorem 1 can be obtained for the
dynamics and the input/output pairs are achieved according to Equation (5). The system internal 
Equation (24) can be written as

\[
\dot{y}_j(t) = \beta_0 y_j(t) x_j(t) + \beta_1 y_j(t) x_{j-1}(t) + \beta_3 y_j(t) x_{j-3}(t),
\]
(29)

where \( \hat{y}_j(t) \) is the approximated output with the second-order Padé approximation and \( \hat{\alpha}_0 y_j(t) = \beta_0 y_j(t) = 12 \tau^{-1} - 2, \hat{\alpha}_1 y_j(t) = 12 \tau^{-2} + 6 \tau^{-1}, \hat{\alpha}_2 y_j(t) = \tau^{-1} + 6 \tau^{-1}, \hat{\beta}_1 y_j(t) = -6 \tau^{-1}, \) and \( \hat{\beta}_2 y_j(t) = \tau^{-1} \) are the delay-dependent coefficients. The relative degree of the third-order system (29) is obtained to be \( r = 1 \) according to Equation (5). The system internal dynamics and the input/output pairs are achieved by choosing \( z_{1j}(t), z_{2j}(t), z_{3j}(t) \)\stackrel{T}{=} [\hat{\beta}_0 y_j(t) x_j(t) + \hat{\beta}_1 y_j(t) x_{j-1}(t) + \hat{\beta}_3 y_j(t) x_{j-3}(t), x_j(t), x_{j-2}(t)]\right. \]. Hence, Equation (24) can be written as

\[
\begin{bmatrix}
\dot{\hat{y}}_{j1}(t) \\
\dot{\hat{y}}_{j2}(t)
\end{bmatrix}
= \begin{bmatrix}
R_{j2}(\tau) & S_{j2}(\tau) \\
P_{j2}(\tau) & Q_{j2}(\tau)
\end{bmatrix}
\begin{bmatrix}
\hat{y}_{j1}(t) \\
\hat{y}_{j2}(t)
\end{bmatrix}
+ \begin{bmatrix}
\beta_{j2}(t) \\
0 \\
0
\end{bmatrix} u_j(t),
\]
(30)

where \( R_{j2}(\tau) = -12 \tau^{-1} - \tau^{-2} \), \( S_{j2}(\tau) = [0, -72 \tau^{-2} \tau^{-1} - 12 \tau^{-2} \tau^{-1}], P_{j2}(\tau) = [0 \tau^{-1}], Q_{j2}(\tau) = [0 \tau^{-2} \tau^{-1}] \) and \( \beta_{j2}(\tau) = \tau^{-1} \).

The unstable eigenvalues for \( j = 2 \) can be obtained similar to the procedure in the preceding subsection as \( \sigma_{1,2} = (3 \pm \sqrt{3}i) \tau^{-1} \), which demonstrate the instability of the internal dynamics (26) for all time delays. By considering a third-order desired error dynamics based on ITAE criterion as \( \tilde{e}(t) + c_2 y_j(t) \tilde{e}(t) + c_1 y_j(t) \tilde{e}(t) + c_0 y_j(t) \tilde{e}(t) = 0 \) with

\[
c_2 y_j(t) = 1.75 \omega_{n_2}(\tau), c_1 y_j(t) = 2.15 \omega_{n_2}(\tau), c_0 y_j(t) = \omega_{n_2}(\tau),
\]
and using Assumption 1 with \( k = \ell = 1 \), the proposed dynamic compensator in Theorem 1 can be obtained for \( j = 2 \) as

\[
\ddot{u}_{n_2}(t) + \gamma_2 y_j(t) \dot{u}_{n_2}(t) + \gamma_1 y_j(t) u_{n_2}(t) = \lambda_0 y_j(t) \tilde{e}(t),
\]
(31)

where \( \lambda_0 y_j(t) = 0.08 \tau^2 \omega_{n_2}(\tau), \gamma_2 y_j(t) = -0.08 \tau^2 \omega_{n_2}(\tau) + 1.75 \omega_{n_2}(\tau) y_j(t), y_j(t) = 0.5 \tau \omega_{n_2}(\tau) + 2.15 \omega_{n_2}(\tau), \) and \( \omega_{n_2}(\tau) \) is the natural frequency of the desired error dynamics that will be determined in the sequel.

The corresponding control law can be achieved using Theorem 2 as

\[
\ddot{u}_j(t) + \gamma_j y_j(t) \dot{u}_j(t) + \gamma_j y_j(t) u_j(t) = \lambda_0 y_j(t) \tilde{e}(t)
\]
\[
= \lambda_0 y_j(t) \tilde{e}(t) + \hat{\alpha}_2 y_j(t) \tilde{e}(t)
\]
\[
+ \hat{\alpha}_0 y_j(t) \tilde{e}(t) + \hat{\alpha}_0 y_j(t) \tilde{e}(t).
\]
(32)

3.3 Performance analysis of the proposed controllers and selection of \( \omega_{n}(\tau) \)

The closed-loop stability of the delayed system (24) is contingent to the delay and its variations. For a time-delay system, it is known that the delay imposes limitations on the system bandwidth (Åström, 2000). However, the Padé approximation is less conservative and leads to a faster response for an actual delay system. Since the phase margin of a time-delay system is lower than its corresponding Padé approximation, this may result in the closed-loop instability. To avoid such a problem in determining the controller parameters, i.e., \( \omega_{n_1}(\tau) \) and \( \omega_{n_2}(\tau) \), we consider the actual delay system (24) rather than the approximated non-minimum phase system. Consider the loop transfer function of the system (24) using the derived controller (28) based on the first-order Padé approximation,

\[
L_j(s) = \frac{\lambda_{0j}(\tau) s^2 + \hat{\alpha}_{j1}(\tau) s + \hat{\alpha}_{j0}(\tau) 1}{s^2 + \gamma_{j1}(\tau) s + \tau s + 1} e^{-\tau s}.
\]
(33)

The corresponding loop transfer function based on the second-order Padé approximation is written as

\[
L_j(s) = \frac{\lambda_{0j}(\tau) s^3 + \hat{\alpha}_{j2}(\tau) s^2 + \hat{\alpha}_{j1}(\tau) s + \hat{\alpha}_{j0}(\tau) 1}{s^3 + \gamma_{j2}(\tau) s^2 + \gamma_{j1}(\tau) s + \tau s + 1} e^{-\tau s}.
\]
(34)

The parameters \( \lambda \) and \( \gamma \) depend on the design parameters \( \omega_{n_1}(\tau) \) and \( \omega_{n_2}(\tau) \), which are varying with respect to delay. Selection of these two parameters depends on the bandwidth limitations imposed by the delay. By considering the phase margin of \( \pi/3 \) and solving Equations (33)
and (34), one can find $\omega_{n1}(\tau)$ and $\omega_{n2}(\tau)$ for each delay value. The resulting points along with a fitted curve have been illustrated in Figure 2 for a sequence of time delay in the interval $0.3 \leq \tau \leq 2.7$ sec with increments of 0.1 sec. It is shown that the second-order Padé results in larger natural frequency for the desired error tracking dynamics thus allowing for faster closed-loop system response. It can be seen that both approximations require lower $\omega_n$ for large delays.

The Bode plots for delay extrema, i.e., $\tau = 0.3$ and 2.7 sec are shown in Figure 3. The first-order Padé approximation yields a cross-over frequency of 1.57 rad/sec for the lowest delay ($\tau = 0.3$ sec) and 0.18 rad/sec for the largest delay ($\tau = 2.7$ sec). The corresponding cross-over frequency for the second-order Padé approximation is 2.31 and 0.26 rad/sec, respectively. Consequently, the second-order approximation exhibits faster response in terms of closed-loop bandwidth compared to the first-order approximation. Although, both approximations provide roll-off of about –20dB/decade, the first-order approximation shows slightly better noise attenuation because of its lower gain characteristics.

Higher-order Padé approximations provide a finite-dimensional solution with fast response to delay systems; however, they increase the sensitivity of the system and thus making it susceptible against perturbations. For the problem of interest, the second-order approximation in Figure 3 exhibits a slight resonant peak, which influences the relative stability of the system. To evaluate the system’s robustness, the Bode plots of the system sensitivity functions have been illustrated in Figure 4. It is shown that by keeping constant phase margin, the second-order Padé results in an increased sensitivity for all delay values. The actual maximum of the sensitivity is $M_s = 1.5$ and 1.9 for the first- and second-order approximation, respectively.

Figure 5 demonstrates the closed-loop response of the actual system versus the approximated non-minimum phase systems using the first and second-order Padé approximation for the largest delay, i.e., $\tau = 2.7$ sec, which indeed is the worst case. The actual system with pure delay in the control-loop $y(t)$ leads to a rise-time of 5.9 sec using the second-order Padé approximation. The corresponding rise-time using the first-order Padé approximation is 10.4 sec, which is considerably larger than the system designed based on the second-order Padé approximation.

3.4 Results and discussion

The experimental data used in this paper are engine air mass flow and engine rpm, which have been collected based on a Federal Test Procedure (FTP) as specified in Figure 6(a) and 6(b). The engine air mass flow is used to obtain the gas transport delay by $\tau_g = \theta / \dot{m}_a$ with $\theta =$
Figure 4. Sensitivity functions of the compensated system with the first and second-order Padé approximation for $0.3 \leq \tau \leq 2.7$ sec with increments of 0.1 sec.

Figure 5. Step response of the actual system with pure delay in the control loop, $y_1(t)$, $y_2(t)$ and the corresponding approximated non-minimum phase systems, $\hat{y}_1(t)$, $\hat{y}_2(t)$ with the first and second-order Padé approximation for $\tau = 2.7$ sec.

Figure 6. (a) Air mass flow for the FTP, (b) engine speed for FTP and (c) estimated time-varying delay, $\tau = \tau_c + \tau_g$. 
1.81 and the engine rpm is used to obtain the cycle delay, \( \tau_c = 120/N \). The overall time-varying delay is depicted in Figure 6(c) for \( 0.3 \text{ sec} \leq \tau \leq 2.7 \text{ sec} \). It is assumed that the engine is commanded to operate at normalised lean AFR, typically 1.1 or 1.4. The simulations are performed in MATLAB/Simulink® using Runge–Kutta ODE4 for numerical integration. The proposed novel controllers (28) and (32) will be used to examine the tracking performance of the closed-loop system.

Figure 8(a) shows the closed-loop system performance in the presence of the time-varying delay, open-loop disturbance profile as shown in Figure 7 and the UEGO sensor measurement noise. The measurement noise is assumed to be a white noise signal with a power intensity of \( 10^{-4} \), which produces a noise with amplitude of 0.05 in the sensor output. The magnified graph within Figure 8(a) shows that for the second-order Padé approximation, the actual system exhibits faster responses compared to the actual system with first-order approximation. The actual system
controlled by the second-order approximation shows 11% overshoot when subject to disturbance at time 100sec (See Figure 7). The closed-loop performance against noise effect has been demonstrated by the magnified graph within Figure 8(b). As it was illustrated through the Bode plots in Figure 3, the closed-loop system is able to attenuate the noise effect for the first-order Padé more efficiently than the second-order approximation. However, for lower delay values, the actual system attenuates well the noise effect for both approximations. This has been shown in the magnified graph within Figure 8(b) for 150sec < t < 160sec (see Figure 6(c)). The controller parameters have been shown in Figures 9 and 10 for the first and second-order Padé approximations, respectively. The parameters $\lambda_0(\tau)$, $\gamma_1(\tau)$, and $\gamma_2(\tau)$ are obtained from Equations (27) and (31) and the other parameters, $\tilde{a}_0(\tau)$, $\tilde{a}_1(\tau)$, and $\tilde{a}_2(\tau)$ are determined from Equations (25) and (29).

4. Conclusion
A dynamic feedback control design strategy was presented for nth-order systems with time-varying delay in the control loop. The jth-order Padé approximation was invoked to tackle the infinite-dimensional problem caused by the pure delay. The configured system within finite-dimensional representation has, however, an unstable internal dynamics because of the utilised approximation. A dynamic compensator was proposed to stabilise the internal dynamics while providing desired tracking error profile. The presented dynamic compensator was then used to design a dynamic controller whose gains were readily calculated based on the desired specifications of the tracking error and dynamic characteristics of the unmatched perturbation. The proposed approach was employed to control a lean-burn spark ignition engine with a time-varying delay in the control loop. The results, which were obtained for different operating conditions, demonstrated the effectiveness of the approach in dealing with disturbances and noisy system measurements. Furthermore, determination of the controller’s parameter-varying gains is straightforward and can be done with simple mathematical operations with no need for look-up tables or scheduling algorithms. Moreover, the proposed approach can be used directly for non-minimum phase systems as well. It is expected that the proposed could lead to an improved fuel economy and emission reduction due to tighter regulation of the AFR. Quantification of the improvements on an experimental test-bed will be pursued in a future study for a Ford F-150 engine. Although the proposed dynamic compensator is shown to be capable of successfully rejecting the effect of unmatched perturbations, it can be evaluated against a broad class of perturbations. The proposed approach can be also extended to nonlinear systems. The tools utilised in the control system design can play a key role in developing such a dynamic controller for nonlinear systems with time-varying delay in the control loop.

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Note
1. The relative degree of a linear system is the difference between the degree of the denominator and numerator of the transfer function.
References


