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Reduced Order Model-based Sliding Mode Control of Dynamic Systems Governed by Burgers' Equation

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Abstract— **In this paper, we use the reduced-order nonlinear model of dynamic systems governed by Burgers' equation with Neumann boundary conditions – recently developed by the authors in [4] – to define low order sliding mode surfaces. While keeping the system states moving on the defined surface, the imposed control law guarantees the stability of the full-order model obtained using a finite element (FE) approximation of the Burgers' equation. The accuracy of the applied reduced-order model obtained from proper orthogonal decomposition (POD) method compared to the FE model is investigated by determining an adequate number of basis functions for the approximating subspace. The reduced-order model is then used to design a sliding mode controller, which is implemented on the FE model demonstrating that the obtained reduced model is suitable for both stabilization of the full-order model and trajectory tracking.**

I. Introduction

The implementation of the standard discretization methods such as finite element or finite difference methods may require a large number of degrees of freedom to accurately describe complex nonlinear partial differential equations (PDEs) like Burgers' equation. Accordingly, the control design task based on those discretized full-order models would be cumbersome. This can also be crucial when the real-time solutions for feedback control of complex systems are sought. As a remedy, the reduced-order modeling was introduced to approximate the original dynamic model by a simpler one so it could still represent certain significant aspects and dominant dynamics of the system or process with an acceptable accuracy depending on the complexity of the reduced-order model. That is to say in implementing different model order reduction schemes, a model with the lowest order, which accurately approximates the original full-order model is desired. To achieve this, the original process should be initially described by a number of basis functions extracted from the expected solution of the system. The development of the model order reduction approaches was proposed in [1] in the framework of the structural simulation and later for simulation of incompressible viscous flows [2].

Recent emergence of data analysis techniques have seen an increased use of tools like principal component analysis (PCA) [3] and proper orthogonal decomposition (POD) for developing control-oriented reduced models of nonlinear systems; a recent example of this is our earlier work [4] that developed a reduced-order model of Burgers' equation using continuous POD. The developed reduced-order model forms the basis for the design of a robust controller in this paper, that is capable of handling possible model uncertainties. Hence, a nonlinear control strategy based on the reduced-order sliding mode is proposed here to tackle different kinds of uncertainties arising from parametric and modeling imprecisions in the reduced-order nonlinear model of Burgers' equation. Sliding mode control (SMC) is a nonlinear feedback control scheme that can effectively apply a high-frequency switching control to alter the dynamics of a nonlinear system [5]. Switching from one continuous mode to another considering the system's current position in state space can guarantee the convergence of the trajectories towards a switching surface that eventually slides along the boundaries of the control structures [6]. For the design of a sliding mode controller, Lyapunov stability theory is utilized to guarantee the stability of the full-order nonlinear model by defining the reduced-order sliding surfaces. The model order reduction methods can be crucial in coping with high order models, in which the measurements of the system states are required. This translates to high computational complexity in both controller and observer design. More specifically, the high dimensional matrix operations including inversion involved in higher order models can considerably increase the run time [7].

In the present work, we employ the POD-based reduced-order model of the Burger's equation developed recently by the authors in [4] for the design of a sliding mode controller that can guarantee the stability of the full-order finite element (FE) model. The closed-loop performance achieved by the reduced-order model based sliding mode controller in the presence of modeling uncertainties demonstrates that the designed nonlinear controller can effectively manipulate the full-order model. This can significantly decrease the computational load to control the high order model.

Throughout the paper, unless otherwise specified, notation $\langle ., . \rangle$ represents the inner product of the given basis functions in the finite element method, which is the spatial domain integration of the product of the given basis functions. Also, $\mathcal{W}_i^l$ represents the i^{th} Fourier coefficient of the reduced-order model of order l and $\mathbb{R}^m$ is the m-dimensional Euclidean space. Moreover, Kronecker delta, δ_{ij}, returns zero for $i \neq j$ and 1 for $i = j$. Finally, we define $A \circ B$ as the Hadamard product of the matrices A and B of the same dimension such that $[A \circ B]_{ij} = [A_{ij}][B_{ij}]$.

The rest of the paper is organized as follows. Section II describes the Burgers' PDE and its finite element modeling and discretization. The continuous POD and the fundamental idea behind it will be reviewed in Section III. The design of a sliding mode controller and the proof of closed-loop system

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stability and reference tracking is given in Section IV. This section also provides a discussion on the observer design for state estimation of the reduced-order model needed for the sliding mode control design. The simulation results are shown in Section V, and the concluding remarks are finally made in Section VI.

II. Approximation of Burgers' Equation with Finite Element Method

In the present study, we consider this nonlinear PDE model with Neumann boundary conditions to develop a reduced order, control-oriented model. To this end, we first approximate this nonlinear PDE model with a large number of ordinary differential equations (ODEs) using finite element models (FEMs), and then reduce it to the state-space form using proper orthogonal decomposition (POD) method.

Suppose that Ω denotes the spatial interval $(0, L)$, and for $T > 0$, we define $Q = (0, T) \times \Omega$. For a given velocity $w(t, x)$ and viscosity ν, the governing viscous Burgers' PDE and the initial and boundary conditions are described by

$$\frac{\partial w(t,x)}{\partial t} + w(t,x)\frac{\partial w(t,x)}{\partial x} - \nu \frac{\partial^2 w(t,x)}{\partial x^2} = f(t,x), \quad (1a)$$

$$\text{I.C.}: \quad w(0,x) = w_0(x), \quad (1b)$$

$$\text{B.C.}: \quad w_x(t,0) = u_1(t), \quad w_x(t,L) = u_2(t), \quad (1c)$$

where $(t, x) \in Q$, $u_1(t)$ and $u_2(t)$ are the varying boundary conditions (i.e., the system inputs) that specify the flux condition on the boundaries. The viscosity is defined as $\nu = \frac{1}{Re}$, where Re represents the Reynolds number. The function f in (1a) is the forcing term assumed to be square integrable in space and time. We define the Hilbert space of Lebesgue square integrable functions as $H = L^2(\Omega)$. Note that the function $f \in H$ if it satisfies

$$\int_0^T \|f(t,x)\|_H^2 \mathrm{d}t < \infty. \quad (2)$$

The finite element method is a powerful tool to approximate PDEs with lumped-parameter ordinary differential equation (ODE) models. An advantage of this technique is that, unlike other methods, if the PDE is time dependent, then it can be reduced to a system of ODEs which can be integrated. Having a system of linear or nonlinear ODEs can allow to represent the model in the linear or nonlinear state-space form, which would be helpful for control synthesis purposes.

FEM Representation of the Burgers' Equation

The FE modeling of Burgers' equation is based on approximating $w(t, x)$ in the space spanned by N piecewise linear basis functions $\mathcal{N}_i(x),\ i = 1, \dots, N$ defined in [4] as

$$w(t,x) = \sum_{i=0}^{N} \mathcal{W}_i(t)\mathcal{N}_i(x), \quad (3)$$

where $\mathcal{W}_i(t)$ is the nodal value at the i^{th} node and time *t*, i.e., $w(t, x_i)$ [8]. The *weak solution* approach is employed here by multiplying both sides of (1a) by a piecewise smooth test function $\mathcal{N}_j(x)$ and integrating in the spatial variable domain [9]. As described in [4], the reduced-order model with the input vector $U(t) = [u_1(t) \;\; u_2(t)]^\top$ is eventually obtained as the following set of $N+1$ ordinary differential equations

$$M\dot{\mathcal{W}}(t) + \nu S\mathcal{W}(t) + N(\mathcal{W}(t)) - \nu LU(t) = F(t), \quad (4)$$

where $\mathcal{W}^2(t) = [\mathcal{W}_0^2(t) \;\dots\; \mathcal{W}_N^2(t)]^\top$ and

$$L = \begin{bmatrix} -1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix}^\top_{(N+1)\times 2},$$

and

$$[M]_{ij} = \langle \mathcal{N}_i(x), \mathcal{N}_j(x) \rangle, \;\; [S]_{ij} = \langle \mathcal{N}_i'(x), \mathcal{N}_j'(x) \rangle,$$
$$F_j(t) = \langle f(t,x), \mathcal{N}_j(x) \rangle, \quad (5)$$

where $(.)'$ denotes the differentiation operator, and

$$N(\mathcal{W}(t)) = \frac{1}{2}K\mathcal{W}^2(t), \quad (6)$$

represents the nonlinear term in the ODEs, where

$$[K]_{ij} = \langle \mathcal{N}_i'(x), \mathcal{N}_j(x) \rangle.$$

The initial condition must be specified in order to solve this set of nonlinear ODEs. To do so, the given initial condition is described in the space spanned by the basis functions, i.e.,

$$w_0(x) \approx w(0,x) = \sum_{i=0}^{N} \mathcal{W}_i(0)\mathcal{N}_i(x). \quad (7)$$

By multiplying the two sides of (7) by the test function $\mathcal{N}_j(x)$ and again employing the weak solution approach, this can be represented in the matrix form as [4]

$$M\mathcal{W}(0) = \mathcal{P}, \quad (8)$$

where $\mathcal{P}_j = \langle w_0(x), \mathcal{N}_j(x) \rangle$. The solution to the linear equation (8) gives the initial conditions needed to solve the set of ODEs in (4).

III. Proper Orthogonal Decomposition Method and its Application to Burgers' Equation

Let $Y = [y_1, \dots, y_n]_{m\times n}$ be a real-valued data matrix containing n temporal snapshot vectors of m spatial data points. The POD basis of rank l is optimal in the sense of representing the columns of Y, i.e., $\{y_j\}_{j=1}^n$, as a linear combination of orthonormal bases of rank l [10]. The optimality is achieved by minimizing the continuous error function between the data and its projection onto the basis set $\{\psi_i\}_{i=1}^l$, $\psi_i \in \mathbb{R}^m$ [11]

$$J = \int_0^T \left\| y(t) - \sum_{i=1}^{l} \langle y(t), \psi_i \rangle_M \psi_i \right\|_M^2 \mathrm{d}t \quad (9)$$
$$\text{s.t.} \quad \langle \psi_i, \psi_j \rangle_M = \delta_{ij} \quad \text{for } 1 \le i, j \le l.$$

As described in [4], the solution to the above constrained optimization problem leads to the following eigenvalue problem

$$\mathcal{R}^n \psi_i = \lambda_i \psi_i \;\; \text{for } i = 1, \dots, l, \quad (10)$$

where the linear, bounded and self-adjoint operator $\mathcal{R}^n : \mathbb{R}^m \to \mathbb{R}^m$ is defined according to the optimality condition in [4].

Next, the derivation of the reduced-order model for the Burgers' equation using POD method is described. To this purpose, we use the approximation of $w(t,x)$ in the space spanned by the POD basis functions $\psi_i(x),\ i=1,\dots,l$, as

$$w(t,x)=\sum_{i=1}^{l}\langle w(t,x),\psi_i(x)\rangle_M\psi_i(x). \tag{11}$$

By setting

$$\mathcal{W}_i^l(t)=\langle w(t,x),\psi_i(x)\rangle_M, \tag{12}$$

we obtain the *Galerkin* projection onto the POD space, that is

$$w(t,x)=\sum_{i=1}^{l}\mathcal{W}_i^l(t)\psi_i(x), \tag{13}$$

where the Fourier coefficients $\mathcal{W}_i^l$, $1\le i\le l$, are functions mapping $[0,T]$ onto $\mathbb{R}$. Since for $l=m$, we have $w(t,x)=w^l(t,x)$, it can be deduced that $w^l(t,x)$ gives an approximation of $w(t,x)$ provided that $l\le m$. In our previous work [4], we showed that the weak solution approach with the piecewise smooth test function $\psi_j(x), j=1,2,\dots,l$ along with the application of POD Galerkin projection would result in the following reduced-order model

$$M^l\dot{\mathcal{W}}^l(t)+\nu S^l\mathcal{W}^l(t)+N^l(\mathcal{W}^l(t))-\nu L^lU=F^l(t), \tag{14}$$

where

$$[M^l]_{ij}=\big\langle\psi_i(x),\psi_j(x)\big\rangle,\qquad [S^l]_{ij}=\big\langle\psi_i'(x),\psi_j'(x)\big\rangle,$$
$$F_j^l(t)=\big\langle f(t,x),\psi_j(x)\big\rangle,$$
$$(N^l(\mathcal{W}^l(t)))_j=\frac{1}{2}\int_0^L N\big(w(t,x)\big)\psi_j(x)\,\mathrm{d}x, \tag{15}$$

and

$$F^l(t)=(\Psi^l)^\top F(t),\ \ L^l=(\Psi^l)^\top L. \tag{16}$$

where $\Psi^l=[\psi_1,\psi_2,\dots,\psi_l]$. It is noted that when the basis functions are orthonormal, $M^l=I_r$ and matrix S^l in the reduced-order model can be obtained from the original full-order matrices by expanding the POD basis functions as

$$S^l=(\Psi^l)^\top S\Psi^l. \tag{17}$$

The POD basis functions can be written as a linear combination of the FE basis functions, and hence

$$\sum_{i=0}^{N}\mathcal{W}_i(t)\mathcal{N}_i(x)=\sum_{i=1}^{l}\mathcal{W}_i^l(t)\sum_{m=0}^{N}\Psi_{mi}\mathcal{N}_m(x). \tag{18}$$

This describes the Fourier coefficients in a compact form as

$$\mathcal{W}(t)=\Psi^l\mathcal{W}^l(t). \tag{19}$$

Substituting (19) into (15) and using Hadamard product notation, we obtain

$$N^l(\mathcal{W}^l(t))=\frac{1}{2}(\Psi^l)^\top K(\Psi^l\mathcal{W}^l(t))\circ(\Psi^l\mathcal{W}^l(t)), \tag{20}$$

and hence the reduced-order model is described by

$$\dot{\mathcal{W}}^l(t)+\nu S^l\mathcal{W}^l(t)+\frac{1}{2}(\Psi^l)^\top K(\Psi^l\mathcal{W}^l(t))\circ(\Psi^l\mathcal{W}^l(t))-\nu L^lU(t)=F^l(t). \tag{21}$$

State-space representation of the reduced-order models

A state-space representation with $N+1$ states for the ODE model obtained from finite element method, i.e., (4), can be determined as

$$\dot{\mathcal{W}}(t)=A\mathcal{W}(t)+\mathbf{h}(t,\mathcal{W}(t),U(t)), \tag{22}$$

where

$$A=-\nu M^{-1}S,$$
$$\mathbf{h}(t,\mathcal{W}(t),U(t))=-\frac{1}{2}M^{-1}K\mathcal{W}^2(t)+M^{-1}F(t)+\nu M^{-1}LU(t).$$

Also, the state-space equivalent of the reduced-order model (21) can be represented as

$$\dot{\mathcal{W}}^l(t)=A^l\mathcal{W}^l(t)+\mathbf{g}(t,\mathcal{W}^l(t),U(t)), \tag{23}$$

where

$$A^l=-\nu S^l,$$
$$\mathbf{g}(t,\mathcal{W}^l(t),U(t))=-\frac{1}{2}(\Psi^l)^\top K(\Psi^l\mathcal{W}^l(t))\circ(\Psi^l\mathcal{W}^l(t))+F^l(t)+\nu L^lU(t).$$

IV. Sliding Mode Control Design Using the Reduced-order Model

The use of a high order controller for real-time control of complex systems is not practical due to the computational complexities involved in both control design process and its implementation. Therefore, the need for a low-order controller is inevitable. The "reduce then design" approach employs the reduced-order state space model given by (23) with an adequately small value of l for control design purposes. In this section, we will explain in detail how a sliding mode controller is designed on the basis of the POD-based reduced order model leading to a stable closed-loop system for the original full-order model.

A. Sliding Mode Control Design

To design a sliding mode controller, a sliding surface should be defined first. To this end, we refer to the state-space representation of the system in (22) accompanied with the full-order system model outputs (or the measurement equations) as

$$\begin{aligned}\dot{\mathcal{W}}(t)&=A\mathcal{W}(t)+\mathbf{h}(t,\mathcal{W}(t),U(t))\\ Y(t)&=C\mathcal{W}(t),\end{aligned} \tag{24}$$

where $\mathcal{W}(t)$ represents the state vector of the full-order model. Also, C represents the system measurement matrix. We assume that only the velocities on the boundaries are measurable, and hence

$$C=\begin{bmatrix}1&0&\dots&0&0\\0&0&\dots&0&1\end{bmatrix}_{2\times(N+1)}.$$

The reduced-order model can be described in the state-space form as

$$\begin{aligned}\dot{\mathcal{W}}^l(t) &= A^l\mathcal{W}^l(t) + \mathbf{g}(t, \mathcal{W}^l(t), U(t)),\\ Y(t) &= C^l\mathcal{W}^l(t),\end{aligned} \tag{25}$$

where $\mathcal{W}^l(t)$ represents the state vector of the reduced-order model. The reduced measurement matrix C^l is obtained by

$$C^l = C\Psi^l. \tag{26}$$

The objective of the control synthesis is to stabilize the closed-loop system and guarantee a reference trajectory tracking while being robust to uncertainties. The sliding surface to ensure that the tracking is eventually achieved is defined on the full-order model as

$$S(t) = Y(t) - r(t) = C\mathcal{W}(t) - r(t), \tag{27}$$

where $S(t) = [S_1(t) \quad S_2(t)]^\top$, $Y(t) = [y_1(t) \quad y_2(t)]^\top$ is the system output and $r(t) = [r_1(t) \quad r_2(t)]^\top$ is the reference signal. The sliding mode controller (SMC) needs an on-line access to the reduced-order model states. Thus, a nonlinear low-order functional observer is designed using the method proposed in [14] to estimate the states of the reduced-order model. The design of the observer will be described at the end of this section. The SMC law usually includes a switching control law and an equivalent control law [5]. A switching control law is employed to drive the system states towards a predefined sliding surface while the equivalent control law guarantees that the system states remain around the sliding surface and converge to the surface. The control law is considered as

$$U(t) = u_{eq}(t) + u_{sw}(t), \tag{28}$$

where $u_{eq}(t)$ and $u_{sw}(t)$ represent the equivalent control and the switching control laws, respectively. To construct the equivalent dynamics, the full-order model is considered while the states are sliding on the defined surface. In fact, only when the system states are on the surface, the equivalent control provides an action. Taking the derivative of (27) results in

$$\begin{aligned}\dot{S} = C\dot{\mathcal{W}}(t) - \dot{r}(t) = C\Big(A\mathcal{W}(t) - \frac{1}{2}M^{-1}K\mathcal{W}^2(t)\\ + M^{-1}F(t) + \nu M^{-1}Lu_{eq}(t)\Big) - \dot{r}(t).\end{aligned} \tag{29}$$

B. Stability Analysis of the Closed-loop System with the Proposed Sliding Model Controller

In order to investigate the asymptotic stability of the closed-loop system with the sliding mode control law, following Lyapunov function is considered

$$V(t) = \frac{1}{2}S^\top S, \tag{30}$$

where S is the sliding surface defined in (27). The stability of the system given by (24) is guaranteed for the sliding surface (27) if

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t} < 0 \quad \text{or} \quad S^\top\dot{S} < 0 \tag{31}$$

in a neighborhood of the surface given by $S(\mathcal{W}) = 0$. Substituting (29) into (31) and using (22), we have

$$\begin{aligned}S^\top\dot{S} &= \big(C\mathcal{W}(t) - r(t)\big)^\top\big(C\dot{\mathcal{W}}(t) - \dot{r}(t)\big)\\ &= \mathcal{W}^\top C^\top C\Big(A\mathcal{W}(t) - \frac{1}{2}M^{-1}K\mathcal{W}^2(t) + M^{-1}F(t)\\ &+ \nu M^{-1}LU(t)\Big) - \mathcal{W}^\top C^\top\dot{r} - r^\top C\Big(A\mathcal{W}(t) + A\mathcal{W}(t)\\ &- \frac{1}{2}M^{-1}K\mathcal{W}^2(t) + M^{-1}F(t) + \nu M^{-1}LU(t)\Big) + r^\top\dot{r}.\end{aligned} \tag{32}$$

As observed from (32), the defined surface is a function of the states of the full-order model. However, the main goal is to implement the control law obtained from the reduced-order model instead of using the full-order one. To this end, the reduced-order model will be used in the defined surface to find the equivalent control law as

$$S(t) = Y(t) - r(t) = C^l\mathcal{W}^l(t) - r(t). \tag{33}$$

The model dynamics while on the sliding surface can be obtained from

$$\dot{S}(t) = \dot{Y}(t) - \dot{r}(t) = C^l\dot{\mathcal{W}}^l(t) - \dot{r}(t) = 0. \tag{34}$$

Solving this equation for u_{eq} by substituting (25) into (34) gives

$$\begin{aligned}u_{eq} = (\nu C^lL^l)^{-1}\Big[\dot{r}(t) - C^l\Big(A^l\mathcal{W}^l(t)\\ - \frac{1}{2}(\Psi^l)^\top K(\Psi^l\mathcal{W}^l(t)) \circ (\Psi^l\mathcal{W}^l(t)) + F^l(t)\Big)\Big].\end{aligned} \tag{35}$$

Finally, substituting (28) and (35) into (32) results in the following

$$\begin{aligned}\frac{\mathrm{d}V(t)}{\mathrm{d}t} = S^\top\Big[C\Big(A\mathcal{W}(t) - \frac{1}{2}M^{-1}K\mathcal{W}^2(t)\\ + M^{-1}F(t)\Big) - (\nu CM^{-1}L)(\nu C^lL^l)^{-1}C^l\Big(A^l\mathcal{W}^l(t)\\ - \frac{1}{2}(\Psi^l)^\top K(\Psi^l\mathcal{W}^l(t)) \circ (\Psi^l\mathcal{W}^l(t)) + F^l(t)\Big)\Big] +\\ S^\top\Big[(\nu CM^{-1}L)(\nu C^lL^l)^{-1} - I_{2\times 2}\Big]\dot{r}(t)\\ + S^\top(\nu CM^{-1}L)u_{sw} < 0.\end{aligned} \tag{36}$$

The latter equation can be represented in a simpler form by rewriting the reduced-order model matrices in terms of the full-order ones using (16) and (26) as

$$\nu C^lL^l = \nu C\Psi^l(\Psi^l)^\top L. \tag{37}$$

On the other hand, the definition of the weighted product and orthonormality of the basis functions leads to

$$M^l = (\Psi^l)^\top M\Psi^l = I_r. \tag{38}$$

After some matrix manipulations, we obtain the following expression for the matrix M,

$$M = \big(\Psi^l(\Psi^l)^\top\big)^{-1}. \tag{39}$$

Combining (37) and (39), we obtain

$$\nu C^l L^l = \nu C\Psi^l(\Psi^l)^\top L = \nu CM^{-1}L. \quad (40)$$

Furthermore, the reduced-order model matrices in (36) can be written in terms of the full-order model matrices using the equations (17), (19) and (26) as

$$C^l\left(A^l\mathcal{W}^l(t)-\frac{1}{2}(\Psi^l)^\top K(\Psi^l\mathcal{W}^l(t))\circ(\Psi^l\mathcal{W}^l(t))+F^l(t)\right)$$
$$= C\Psi^l(-\nu(\Psi^l)^\top S\Psi^l)\mathcal{W}^l(t)$$
$$-\frac{1}{2}C\Psi^l(\Psi^l)^\top K(\Psi^l\mathcal{W}^l(t))\circ(\Psi^l\mathcal{W}^l(t)) + C\Psi^l(\Psi^l)^\top F(t)$$
$$= C(-\nu M^{-1}S)\mathcal{W}(t)+\frac{1}{2}CM^{-1}K\mathcal{W}(t)\circ\mathcal{W}(t)+CM^{-1}F(t)$$
$$= C\left(A\mathcal{W}(t)-\frac{1}{2}M^{-1}K\mathcal{W}^2(t)+M^{-1}F(t)\right). \quad (41)$$

By substituting (41) back into (36), we have

$$S^\top(\nu CM^{-1}L)u_{sw} < 0, \quad (42)$$

which implies that the switching control law only needs to satisfy the inequality condition (42). Considering (40) and the fact that the system under study is a multi-input multi-output system, the switching control law corresponding to the defined surfaces can be written as

$$u_{sw}(t) = -(\nu C^l L^l)^{-1}\begin{bmatrix} \lambda_1 S_1 + \xi_1 sat(S_1) \\ \lambda_2 S_2 + \xi_2 sat(S_2) \end{bmatrix}, \quad (43)$$

where λ_1, λ_2, ξ_1 and ξ_2 are positive constants and $sat(.)$ is the saturation function with the upper limit of 1 and lower limit of -1. These constants, which are chosen by trial and error considering the trade-off between the reaching time and chattering, can be considered large enough when the trajectory is far from the switching surface (so that the reaching time is short), and then as small as desired in order to limit the chattering.

C. Functional Observer Design

A functional observer can be designed to estimate the states of the reduced-order model instead of the full-order one. Hence, the computational cost would be much lower compared to the full-order state observer. The reduced-order model described by (23) is obtained from the discretized Burgers' equation with a locally Lipschitz nonlinearity with respect to $\mathcal{W}^l$ in a region $\mathcal{D}$, i.e., for any $\mathcal{W}_1^l(t), \mathcal{W}_2^l(t) \in \mathcal{D}$ [14]

$$\|\mathbf{g}(\mathcal{W}_1^l, U^*) - \mathbf{g}(\mathcal{W}_2^l, U^*)\| \leq \gamma_d\|\mathcal{W}_1^l - \mathcal{W}_2^l\|, \quad (44)$$

where $\| \,.\, \|$ represents the induced 2-norm, U^* is an admissible control sequence and γ_d is the nonnegative Lipschitz constant. If the nonlinear function $\mathbf{g}(.,.)$ globally satisfies the Lipschitz continuity condition in $\mathbb{R}^l$, then the global stability of the observer is guaranteed [14]. The proposed observer takes the following form

$$\dot{\hat{\mathcal{W}}}^l(t) = A^l\hat{\mathcal{W}}^l(t) + \mathbf{g}(\hat{\mathcal{W}}^l(t), U(t)) + L(Y(t) - C^l\hat{\mathcal{W}}^l(t)), \quad (45)$$

where $\hat{\mathcal{W}}^l$ is the estimate of $\mathcal{W}^l$ and L is the observer matrix gain chosen such that the observer error system is asymptotically stable. As shown in [14], a sufficient linear matrix inequality (LMI) condition can be determined to maximize γ_d while a stabilizing L is obtained. The observer of order l in (45) is used to estimate $\mathcal{W}^l$ considering that the quadratic nonlinearities in the reduced-order model described by (23) are locally Lipschitz.

V. Simulation Results and Discussion

In this section, we illustrate some of the results of our numerical studies and further provide a discussion on the accuracy of the derived reduced-order models, as well as the high performance of the closed-loop system achieved from the implementation of the designed sliding mode controller.

A. Open-loop System Simulation Results to Examine the Reduced-oder Model Accuracy

In order to assess the performance of the presented model reduction method, an example of a viscous Burgers' equation is examined here. The forcing term in (1a) is considered to be zero, which translates to the so-called viscous Burgers' equation. The initial condition is assumed to be

$$w_0(x) = \begin{cases} 100(sin(8\pi x) - 2x), & \text{for } x \in (0, \frac{1}{4}] \\ 0, & \text{otherwise.} \end{cases}$$

It is also assumed that the boundary conditions, i.e., inputs to the state-space models, are sinusoidal functions as

$$u_1(t) = 0.8sin(2t), \;\; u_2(t) = 0.5sin(t). \quad (46)$$

The reduced and full-order open-loop models are simulated for a given viscosity $\nu = 0.01$ (or *Re* = 100) to gauge the performance of the model reduction approach for a class of physical flows. It is observed that by increasing the number of basis functions to 7, a very close match between the outputs of the two models can be achieved (see [4]). Finally, to quantify the model accuracy, we consider the *Best Fit Rate* (BFR) as implemented in [4].

TABLE I
THE MSE AND BFR OF THE OUTPUT SIGNAL OF THE REDUCED ORDER MODELS WITH SINUSOIDAL INPUTS (46)

	POD (3 Bases)		POD (5 Bases)		POD (7 Bases)	
Output	MSE	BFR	MSE	BFR	MSE	BFR
y_1	0.0021	0.0886	2.1836e-04	0.7042	1.3275e-05	0.9271
y_2	1.0203e-04	0.4909	6.7584e-06	0.8690	4.2925e-06	0.8956

B. Closed-loop Simulation Results with the Designed Sliding Mode Controller

The discretized full-order model obtained from FEM is used to validate the designed sliding mode controller (SMC) consisting of equivalent and switching control laws in tracking a given reference trajectory. From the fluid mechanics point of view, this can be seen as the problem of controlling the flux on the boundaries to reach the desired flow velocity at the desired points. The viscous Burgers' equation with the same initial condition as given in the previous section is used to validate the proposed SMC design approach. The two components of the SMC laws are obtained from the 7^{th}

order reduced model. As described before, the measurement devices are considered to be placed on the boundaries to collect the flow velocity as the system output. A sinusoidal signal is considered as the reference input and the system output and control input are shown for the reference input. The tracking performance and control inputs for a given

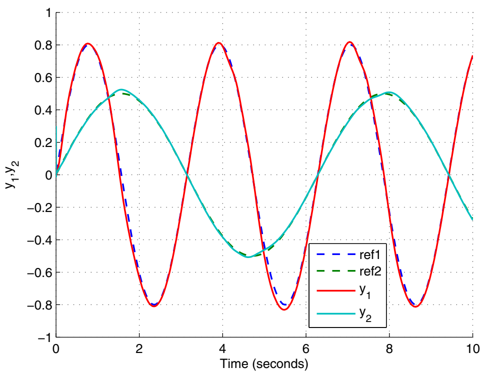

Fig. 1. System outputs and the reference inputs for the given sinusoidal function.

sinusoidal reference signal are shown in Figures 1 and 2. The corresponding sliding mode controller parameters are tuned as $\lambda_1 = 1.3 \times 10^4$, $\lambda_2 = 1.95 \times 10^4$, $\xi_1 = 1.4 \times 10^3$ and $\xi_2 = 1.3 \times 10^3$. The simulation result of the full-order model is shown in Figure 3. As observed from Figure 1, the proposed SMC law illustrates a high tracking performance in the presence of model uncertainties. Uncertainties in the problem in hand are primarily due to the discrepancy between the full-order and reduced-order models, where the number of the eigenfunctions chosen to find the POD bases dictates the level of uncertainties. The switching control in the SMC law works in favor of keeping the trajectory on the defined sliding surface in the presence of the aforedescribed uncertainties.

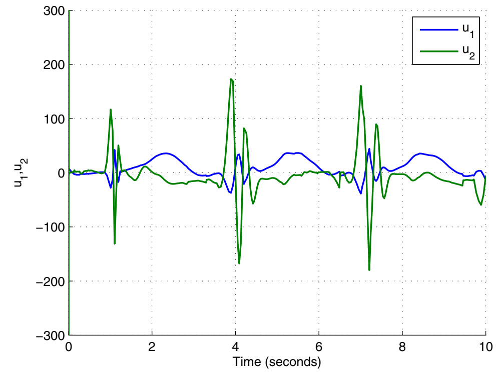

Fig. 2. The control inputs from the proposed SMC law to track a sinusoidal function.

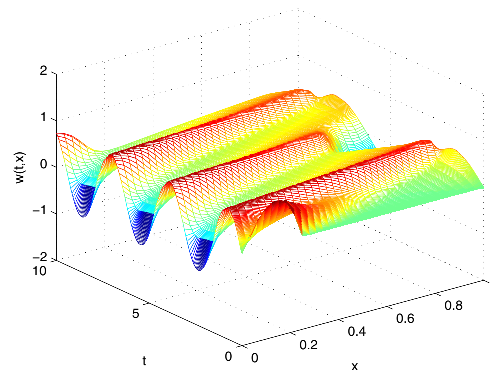

Fig. 3. Velocity response of the full-order model with the proposed SMC law to track a sinusoidal function.

VI. Concluding Remarks

In this paper, the developed reduced-order model in [4] has been used for the design of a sliding mode controller on the basis of the sliding surfaces defined according to the reduced-order model. Due to the need for state estimates required by the sliding mode controller, implementing the reduced-order model significantly decreased the computational load for both controller and observer design. Finally, numerical studies have demonstrated promising results by using the proposed reduced-order model and controller to achieve a high performance tracking of different reference trajectories. This overall proved the practicality of the proposed control-oriented modeling and model-based nonlinear control design approach for complex systems governed by nonlinear PDEs.

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