51st IEEE Conference on Decision and Control
December 10-13, 2012. Maui, Hawaii, USA

Sampled-Data Control of LPV Systems Using Input Delay Approach

Amin Ramezanifar†⋆, Javad Mohammadpour‡, Karolos Grigoriadis†

***Abstract*— In this paper, we address the problem of sampled-data control design for continuous-time linear parameter varying (LPV) systems. The state-feedback control design method we develop in this paper relies on the information from the sampled states. Due to the combination of the system continuous-time dynamics and the controller discrete-time dynamics connected through A/D and D/A converter devices, the closed-loop system is a hybrid one. In order to analyze this hybrid system from both stability and performance perspectives, we employ the so-called *input delay* approach to map the closed-loop system into the continuous-time domain. This would help in the design of a sampled-data controller in a direct way without actually discretizing the continuous-time system. Due to the state-delay representation of the transformed closed-loop system imposed by the sampled-data controller, we further develop a delay-dependent approach for control synthesis purposes. The simulation results demonstrate the viability of the proposed control design method to satisfy the stability and performance objectives even for the varying sampling case.**

I. INTRODUCTION

Linear parameter varying (LPV) system theory has led to the significant improvements in the study of time-varying and nonlinear systems. LPV systems include a class of linear systems, whose dynamics depends on time-varying parameters. The problem of sampled-data control for continuous-time systems has been a matter of great importance in engineering systems and is more challenging for LPV systems than for linear time invariant (LTI) systems. Digital control of analog systems yields a closed-loop system with hybrid configuration that contains both continuous- and discrete-time signals and is difficult to handle mathematically. A particular difficulty, that has been the main concern of researchers in this area, is to ensure that the digital controller meets the design specifications in between the samples [4]. One well-established approach in this category is the lifting method presented in [1]. This approach addresses the sampled-data problem in terms of an equivalent discrete-time system, where the plant is first augmented by the converter devices and then lifted to a system with a finite dimensional state-space and infinite dimensional input and output spaces. Using the lifting technique, the authors in [13] addressed the problem of sampled-data control design for LPV systems. Unfortunately, lifting method cannot be applied to the uncertain sampling system or uncertain system matrices [6] and also is computationally complicated. An alternative direct method proposed in [8] and further studied in [6], [7], [12] copes with the hybrid nature of sampled-data control systems by reformulating the digital control law as a delayed continuous-time one, that is

$$u(t) = u_d(t_k) = u_d(t-(t-t_k)) = u_d(t-\tau(t)) \quad t_k \le t < t_{k+1} \quad (1)$$

where $\tau(t) = t - t_k$ denotes a time-varying delay. By taking advantage of this idea, the closed-loop system is mapped to the continuous-time domain, to which the recent work on control design for systems with delay can be applied.

The literature on stability analysis and control of time-delay systems is rich (see [9], [10] and numerous references therein). The existing criteria for analysis of time delay systems are categorized into either delay-independent or delay-dependent approaches [9]. In the delay-independent approach, a controller is designed such that the system remains stable regardless of the time delay magnitude. In contrast, by considering the information of the time delay, the delay-dependent approach leads to generally less conservative results specifically for smaller time delays. One of the issues with the input delay approach for sampled-data control is that the time delay (as formulated in (1)) itself is time varying. This implies that the sampled-data control using input delay approach leads to a time delay LPV system even if the open-loop is a delay-free system. Due to the time delay LPV nature of the resulting system, one can use the recent results on analysis and control synthesis of time delay LPV systems that has been of interest to many researchers in controls community in the past decade (see, [2], [15], [16]).

Contribution of the present paper is as follows. We propose a method for the design of sampled-data state feedback controllers for LPV systems. The design should guarantee asymptotic stability and a specified level of performance on the closed-loop hybrid system. The main result of this paper is inspired by the authors' recent work in [18] and the so-called input delay method for sampled-data control design proposed in [7]. In particular, we use a parameter-dependent Lyapunov-Krasovskii functional that results in a delay-dependent synthesis method that, unlike the existing approaches in the literature, can handle fast-varying time delay ($\dot{\tau} \ge 1$). To ensure that the solution to the synthesis problem is in the form of a linear matrix inequality (LMI) optimization problem, we introduce slack variables (see [14]) to relax the resulting condition in terms of an LMI problem. Using the derived formulation based on the slack variables, we then obtain the synthesis condition for the state-feedback sampled-data control design that is considerably simpler compared to the existing methods.

The notation used in this paper is standard. $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+$ is the set of non-negative real

†Dept. of Mechanical Engineering, University of Houston, Houston, TX 77204
‡College of Engineering, University of Georgia, Athens, GA 30602
⋆Corresponding author, E-mail: aramezan@mail.uh.edu

978-1-4673-2066-5/12/$31.00 ©2012 IEEE

numbers. $\mathbb{R}^n$ and $\mathbb{R}^{k\times m}$ are used to denote the set of real vectors of dimension n and the set of real $k\times m$ matrices, respectively. In addition, $\mathbb{S}^{n\times n}$ and $\mathbb{S}_+^{n\times n}$ denote the set of real symmetric $n\times n$ matrices and symmetric positive definite matrices, respectively. In a symmetric matrix, the asterisk $\star$ in the (i,j) element denotes transpose of (j,i) element.

II. Preliminaries and Problem Statement

We consider the following state-space representation for a linear parameter varying (LPV) system

$$\begin{aligned}\dot{x}(t) &= A(\rho(t))x(t)+B_1(\rho(t))w(t)+B_2(\rho(t))u(t)\\ z(t) &= C_1(\rho(t))x(t)+D_{11}(\rho(t))w(t)+D_{12}(\rho(t))u(t), \quad (2)\end{aligned}$$

where $x(t)\in\mathbb{R}^n$ is the state vector, $z(t)\in\mathbb{R}^{n_z}$ is the vector of controlled outputs, $w(t)\in\mathbb{R}^{n_w}$ is exogenous disturbance vector containing both process and measurement noise with finite energy and $u(t)\in\mathbb{R}^{n_u}$ is the control input vector. The system matrices $A(\cdot)$, $B_1(\cdot)$, $B_2(\cdot)$, $C_1(\cdot)$, $D_{11}(\cdot)$ and $D_{12}(\cdot)$ are real continuous functions of a time varying parameter vector $\rho(t)\in\mathscr{F}_{\mathscr{P}}^{\nu}$ and of appropriate dimensions, where $\mathscr{F}_{\mathscr{P}}^{\nu}$ is the set of allowable parameter trajectories defined as

$$\mathscr{F}_{\mathscr{P}}^{\nu}\equiv\{\rho:\rho(t)\in C(\mathbb{R},\mathbb{R}^s):\rho(t)\in\mathscr{P},|\dot{\rho}_i(t)|\le\nu_i \quad i=1,2,...,s,\ \forall t\in\mathbb{R}_+\} \quad (3)$$

where $C(\mathbb{R},\mathbb{R}^s)$ is the set of continuous-time functions from $\mathbb{R}$ to $\mathbb{R}^s$, $\mathscr{P}$ is a compact set of $\mathbb{R}^s$, and $\{\nu_i\}_{i=1}^s$ are nonnegative numbers. The constraints in (3) imply that the parameter trajectories and their variations are bounded. In the present paper, we are interested in designing a sampled-data state-feedback controller that uses the discrete samples of system states as input and provides control input to the plant by holding the controller's discrete output. Shown in Fig. 1 is the configuration of the closed-loop system. For

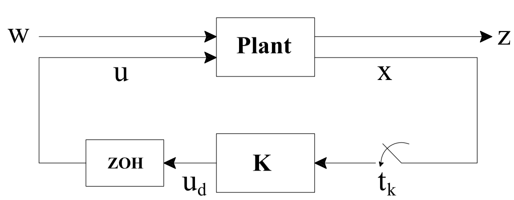

Fig. 1. Sampled data control of a continuous-time system

control design purposes, we assume that the system states are fully available and are sampled by an ideal sampler (see Fig. 1). We further assume that the sampling instants are not necessarily equi-spaced but are constrained to $t_{k+1}-t_k\le h$ for two consecutive samples k and $k+1$. The controller output at each sampling instant is updated based on the value of the state vector and corresponding LPV parameter vector. That is,

$$u_d(t_k)=K(\rho(t_k))x(t_k). \quad (4)$$

Using a zero-order hold (ZOH shown in Fig. 1), the controller output becomes a piecewise-constant signal and is fed to the continuous-time plant, *i.e.*,

$$u(t)=u_d(t_k) \quad t_k\le t<t_{k+1}. \quad (5)$$

The design has to ensure the stability of the closed-loop hybrid system and leads to an optimal performance measured in terms of the induced $\mathscr{L}_2$ gain (or $\mathscr{H}_\infty$ norm) of the closed-loop system. This measure is used to minimize the worst-case energy of the output $z(t)$ over all bounded energy disturbances $w(t)$ in Fig. 1 defined by

$$\|T_{wz}\|_{i,2}=\sup_{\rho\in\mathscr{F}_{\mathscr{P}}^{\nu}}\ \sup_{w\in\mathscr{L}_2-\{0\}}\frac{\|z\|_{\mathscr{L}_2}}{\|w\|_{\mathscr{L}_2}} \quad (6)$$

where T_{wz} is the operator mapping the input vector $w(t)$ to the output vector $z(t)$. The γ-suboptimal $\mathscr{H}_\infty$ control problem seeks for a controller that yields $\|T_{wz}\|_{i,2}<\gamma$ for some positive scalar value γ. One might rather state that the condition $\|T_{wz}\|_{i,2}<\gamma$ guarantees that the output vector energy will be bounded by $\gamma\|w\|_{\mathscr{L}_2}$ for all possible bounded energy disturbances $w(t)$. Due to the hybrid nature of the closed-loop system determined from the aforedescribed sampled-data control problem, representation of the closed-loop system in a unified domain is difficult. In this paper, we use the input delay approach based on (1) to represent the digital control law as a continuous time-varying delay. The time-varying delay is bounded as $\tau\le t_{k+1}-t_k\le h$ and is also piecewise linear with the rate of variation $\frac{d\tau}{dt}=1$ for $t\ne t_k$. The closed-loop system interconnection of (2), (4) and (5) is

$$\begin{aligned}\dot{x}(t) &= Ax(t)+B_2K(\rho(t-\tau(t)))x(t-\tau(t))+B_1w(t)\\ z(t) &= C_1x(t)+D_{12}K(\rho(t-\tau(t)))x(t-\tau(t))+D_{11}w(t). \quad (7)\end{aligned}$$

By introducing A_h and C_{1h} as

$$\begin{aligned}A_h(\rho(t)) &= B_2(\rho(t))K(\rho(t-\tau(t)))\\ C_{1h}(\rho(t)) &= D_{12}(\rho(t))K(\rho(t-\tau(t))), \quad (8)\end{aligned}$$

the following parameter-dependent delayed state-space representation is obtained

$$\begin{aligned}\dot{x}(t) &= A(\rho(t))x(t)+A_h(\rho(t))x(t-\tau(t)))+B_1(\rho(t))w(t)\\ z(t) &= C_1(\rho(t))x(t)+C_{1h}(\rho(t))x(t-\tau(t))+D_{11}(\rho(t))w(t). \quad (9)\end{aligned}$$

In summary, the interconnection of the open-loop system (2) and the controller (5) is represented as a continuous-time LPV state delay system using the input delay approach.

Next, we provide some useful lemmas that will play a key role in the proofs of the main results of the paper.

Lemma 1 [17] (Cauchy-Schwartz Inequality): For any positive definite matrix P and any $\nu(\alpha)\in\mathbb{R}^n$

$$h\int_{t-h}^{t}\nu(\alpha)^TP\nu(\alpha)d\alpha\ge\left[\int_{t-h}^{t}\nu(\alpha)d\alpha\right]^TP\left[\int_{t-h}^{t}\nu(\alpha)d\alpha\right].$$

Lemma 2 [11] (Projection Lemma): Given a symmetric matrix $\Psi\in\mathbb{R}^{m\times m}$ and two matrices Λ and Γ of appropriate dimensions, the linear matrix inequality

$$\Psi+\Lambda^T\Theta^T\Gamma+\Gamma^T\Theta\Lambda<0 \quad (10)$$

is feasible in matrix Θ if and only if

$$\mathcal{N}_\Lambda^T \Psi \mathcal{N}_\Lambda < 0 \tag{11}$$

and

$$\mathcal{N}_\Gamma^T \Psi \mathcal{N}_\Gamma < 0 \tag{12}$$

where $\mathcal{N}_\Lambda$ and $\mathcal{N}_\Gamma$ are the corresponding null spaces.

III. Stability and Performance Analysis of Time-delay LPV Systems

In this section, we present stability and $\mathcal{H}_\infty$ norm performance analysis conditions for time-delay LPV systems by deriving a set of linear matrix inequality (LMI) problems. We note that the existing LMI-based results in the literature on analysis of time-delay LPV systems are only applicable to the systems, in which the time-delay satisfies the condition $\dot{\tau} < 1$. For the sampled-data control problem under study in this paper, we have $\dot{\tau} = 1$ and hence a set of new conditions must be developed.

A. Stability Analysis

We first consider the following unforced LPV system with the state delay

$$\dot{x}(t) = A(\rho(t))x(t) + A_h(\rho(t))x(t-\tau(t)). \tag{13}$$

Lyapunov-Krasovskii stability theory serves as a useful tool to achieve delay-dependent conditions for the stability analysis of the system represented by (13). To this aim, we need to find a positive definite functional with an infinitesimal upper bound, whose time derivative is negative. The interested reader is referred to [5], [9], [17] for an extensive review of the theory and the Lyapunov-Krasovskii functional selection. As the first result of this paper, we present the following theorem as a sufficient condition to ensure asymptotic stability of the LPV system (13).

Theorem 1: The time-delay LPV system (13) is asymptotically stable for all $\tau(t) \le h$ if there exist a continuously differentiable matrix function $P : \mathbb{R}^s \to \mathbb{S}_+^{n\times n}$ and a constant matrix $R \in \mathbb{S}_+^{n\times n}$ such that for all $\rho(t) \in \mathcal{F}_\mathcal{P}^\nu$, there is a feasible solution to the following LMI problem

$$\begin{bmatrix} A^TP + PA + \dot{P} - R & PA_h + R & hA^TR \\ \star & -R & hA_hR \\ \star & \star & -R \end{bmatrix} < 0. \tag{14}$$

Proof. We consider the following Lyapunov-Krasovskii functional

$$V(x_t,\rho) = x^T(t)P(\rho(t))x(t) + \int_{-h}^{0}\int_{t+\theta}^{t} \dot{x}^T(\xi)hR\dot{x}(\xi)\,d\xi d\theta, \tag{15}$$

where P and R are as defined in Theorem 1 and the notation $x_t(\theta)$ is used to represent $x(t+\theta)$ for $\theta \in [-h \quad 0]$. It can be shown that (15) is a positive definite with infinitesimal upper bound functional. It is noted that (15) is chosen to be dependent on the LPV parameter vector $\rho(t)$ and the maximum sampling interval h to result in less conservative stability conditions [3]. In order for the system (13) to be asymptotically stable, it suffices that time derivative of (15) along the trajectories of the system (13) is negative definite. We have

$$\dot{V}(x_t,\rho) = \dot{x}^TPx + x^TP\dot{x} + x^T\dot{P}x + h^2\dot{x}^TR\dot{x} - \int_{t-h}^{t}\dot{x}^T(\theta)hR\dot{x}(\theta)d\theta. \tag{16}$$

Since $\tau(t) \le h$, the integral term in the above satisfies

$$-\int_{t-h}^{t}\dot{x}^T(\theta)hR\dot{x}(\theta)\,d\theta \le -\int_{t-\tau}^{t}\dot{x}^T(\theta)hR\dot{x}(\theta)d\theta.$$

Employing Lemma 1, we can bound the right hand side of the above inequality by

$$\begin{aligned} -\int_{t-\tau}^{t}\dot{x}^T(\theta)hR\dot{x}(\theta)\,d\theta &\le -\frac{h}{\tau}\left(\int_{t-\tau}^{t}\dot{x}^T(\theta)\,d\theta\right)^T R\left(\int_{t-\tau}^{t}\dot{x}^T(\theta)\,d\theta\right) \\ &= -\frac{h}{\tau}[x(t)-x(t-\tau(t))]^T R[x(t)-x(t-\tau(t))] \end{aligned}$$

Since $-\frac{h}{\tau} \le -1$, the following inequality is obtained

$$-\int_{t-h}^{t}\dot{x}^T(\theta)hR\dot{x}(\theta)\,d\theta \le -[x(t)-x(t-\tau(t))]^T R[x(t)-x(t-\tau(t))] \tag{17}$$

Substituting (17) in (16), we obtain

$$\begin{aligned} \dot{V}(x_t,\rho) \le\ & \dot{x}^TPx + x^TP\dot{x} + x^T\dot{P}x + h^2\dot{x}^TR\dot{x} \\ & - [x(t)-x(t-\tau(t))]^T R[x(t)-x(t-\tau(t))]. \end{aligned} \tag{18}$$

Further simplification and collecting the terms in (18) yields

$$\begin{aligned} \dot{V}(x_t,\rho) &\le \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix}^T \left(\mathcal{X} + \begin{bmatrix} A^T \\ A_h^T \end{bmatrix} h^2R \begin{bmatrix} A^T \\ A_h^T \end{bmatrix}^T\right) \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix} \\ &= \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix}^T \left(\mathcal{X} + \begin{bmatrix} hA^TR \\ hA_h^TR \end{bmatrix} R^{-1} \begin{bmatrix} hA^TR \\ hA_h^TR \end{bmatrix}^T\right) \begin{bmatrix} x(t) \\ x(t-\tau) \end{bmatrix}, \end{aligned} \tag{19}$$

where

$$\mathcal{X} = \begin{bmatrix} A^TP + PA + \dot{P} - R & PA_h + R \\ \star & -R \end{bmatrix}.$$

To ensure that $\dot{V}(x_t,\rho) < 0$ using (19), it is sufficient that

$$\begin{bmatrix} A^TP + PA + \dot{P} - R & PA_h + R \\ \star & -R \end{bmatrix} + \begin{bmatrix} hA^TR \\ hA_h^TR \end{bmatrix} R^{-1} \begin{bmatrix} hA^TR \\ hA_h^TR \end{bmatrix}^T < 0.$$

Finally, applying Schur complement to the above LMI results in the condition (14), and this completes the proof. ■

B. Performance Analysis

In this part, we present the performance analysis condition for the time-delay LPV system (9). The derived condition which is in LMI form will be used in the next section for sampled-data control design.

Theorem 2: The LPV system (9) is asymptotically stable and the condition $\|z\|_{\mathcal{L}_2} \le \gamma\|w\|_{\mathcal{L}_2}$ holds true for $\tau(t) \le h$ and zero initial condition if there exist a continuously differentiable matrix function $P : \mathbb{R}^s \to \mathbb{S}_+^{n\times n}$, a constant matrix $R \in \mathbb{S}_+^{n\times n}$ and a positive scalar γ such that

$$\begin{bmatrix} A^TP + PA + \dot{P} - R & PA_h + R & PB_1 & C_1^T & hA^TR \\ \star & -R & 0 & C_{1h}^T & hA_h^TR \\ \star & \star & -\gamma I & D_{11}^T & hB_1^TR \\ \star & \star & \star & -\gamma I & 0 \\ \star & \star & \star & \star & -R \end{bmatrix} < 0 \tag{20}$$

Proof: We first define a Lyapunov-Krasovskii functional similar to the one introduced in Theorem 1. Next, we apply the following congruent transformation

$$\mathscr{T} = \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \end{bmatrix}$$

to (20). In the obtained inequality, it can be observed that the negative definiteness of the upper left 3×3 block matrix, in light of Theorem 1, concludes the asymptotical stability of the system (9). Applying Schur complement to (20) results in

$$\begin{bmatrix} A^TP+PA+\dot{P}-R & PA_h+R & PB_1 & C_1^T \\ \star & -R & 0 & C_{1h}^T \\ \star & \star & -\gamma I & D_{11}^T \\ \star & \star & \star & -\gamma I \end{bmatrix} + \begin{bmatrix} A^T \\ A_h^T \\ B_1^T \\ 0 \end{bmatrix} h^2 R \begin{bmatrix} A^T \\ A_h^T \\ B_1^T \\ 0 \end{bmatrix}^T < 0. \quad (21)$$

By another use of Schur complement on (21), we obtain

$$\begin{bmatrix} A^TP+PA+\dot{P}-R & PA_h+R & PB_1 \\ \star & -R & 0 \\ \star & \star & -\gamma I \end{bmatrix} + \begin{bmatrix} C_1^T \\ C_{1h}^T \\ D_{11}^T \end{bmatrix} \gamma^{-1} \begin{bmatrix} C_1^T \\ C_{1h}^T \\ D_{11}^T \end{bmatrix}^T + \begin{bmatrix} A^T \\ A_h^T \\ B_1^T \end{bmatrix} h^2 R \begin{bmatrix} A^T \\ A_h^T \\ B_1^T \end{bmatrix}^T < 0.$$

Multiplying the above inequality from left and right by $[x^T(t) \quad x^T(t-\tau) \quad w^T(t)]^T$ and its transpose respectively and further algebraic manipulations, we obtain

$$\begin{aligned} &\dot{x}^TPx + x^TP\dot{x} + x^T\dot{P}x - h^2\dot{x}^TR\dot{x} \\ &-[x(t)-x(t-\tau(t))]^T R\,[x(t)-x(t-\tau(t))] \\ &-\gamma w^T(t)w(t) + \frac{1}{\gamma} z^T(t)z(t) < 0, \end{aligned}$$

and using (18), we have

$$\dot{V}(x_t,\rho) - \gamma w^T(t)w(t) + \frac{1}{\gamma} z^T(t)z(t) < 0. \quad (22)$$

Integrating both sides of the inequality (22) from 0 to ∞ and using $V|_{t=0} = V|_{t=\infty} = 0$ (due to the asymptotical stability and zero initial condition), we arrive at $\|z\|_{\mathscr{L}_2} \le \gamma \|w\|_{\mathscr{L}_2}$ and this completes the proof. ■

IV. LMI Relaxation Using Slack Variables

We first describe the motivation behind the use of slack variables in this paper. In the design of the sampled-data controller, the goal is to establish a synthesis condition to ensure that (7) is stable and satisfies a prescribed level of $\mathscr{H}_\infty$ performance. To this purpose, the corresponding system matrices are substituted in (20); this, however, results in a bilinear matrix inequality problem due to the byproduct of the controller matrix K with the unknown matrix function P and matrix R. Therefore, we will seek an alternative method based on the introduction of *slack variables* to reformulate the corresponding problem to ensure that an LMI is achieved.

The following theorem provides an alternative way to solve the matrix inequality (20).

Theorem 3: The LPV system (9) is asymptotically stable for all $\tau(t) \le h$ and satisfies $\|z\|_{\mathscr{L}_2} \le \gamma \|w\|_{\mathscr{L}_2}$ if there exist a continuously differentiable matrix function $P: \mathbb{R}^s \to \mathbb{S}_+^{n\times n}$, constant matrices R, V_1, V_2, $V_3 \in \mathbb{S}_+^{n\times n}$ and a positive scalar γ such that for any admissible parameter trajectory $\rho(t) \in \mathscr{F}_{\mathscr{P}}^{\nu}$, the following LMI problem has a feasible solution

$$\begin{bmatrix} -V_1-V_1^T & P-V_2^T+V_1A & -V_3^T+V_1A_h & V_1B_1 & 0 & hR \\ \star & \dot{P}-R+A^TV_2^T+V_2A & R+A^TV_3^T+V_2A_h & V_2B_1 & C_1^T & 0 \\ \star & \star & -R+A_h^TV_3^T+V_3A_h & V_3B_1 & C_{1h}^T & 0 \\ \star & \star & \star & -\gamma I & D_{11}^T & 0 \\ \star & \star & \star & \star & -\gamma I & 0 \\ \star & \star & \star & \star & \star & -R \end{bmatrix} < 0. \quad (23)$$

Proof: We rewrite (23) as $\Psi + \Lambda^T\Theta^T\Gamma + \Gamma^T\Theta\Lambda < 0$, with

$$\Psi = \begin{bmatrix} 0 & P & 0 & 0 & 0 & hR \\ \star & \dot{P}-R & R & 0 & C_1^T & 0 \\ \star & \star & -R & 0 & C_{1h} & 0 \\ \star & \star & \star & -\gamma I & D_{11}^T & 0 \\ \star & \star & \star & \star & -\gamma I & 0 \\ \star & \star & \star & \star & \star & -R \end{bmatrix},$$

$$\Lambda = \begin{bmatrix} -I & A & A_h & B_1 & 0 & 0 \end{bmatrix},$$

and

$$\Gamma = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \end{bmatrix}, \quad \Theta = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}. \quad (24)$$

The matrix variables V_1, V_2 and V_3 are known as slack variables [14]. We next use Lemma 2 (Projection Lemma) by finding the bases for the null space of Λ and Γ as

$$\mathscr{N}_\Lambda = \begin{bmatrix} A & A_h & B_1 & 0 & 0 \\ I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix}, \quad \mathscr{N}_\Gamma = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}.$$

We then substitute the two matrices above in the solvability conditions of Lemma 2. Using the solvability condition (11) results in the LMI condition (23). On the other hand, the second solvability condition, *i.e.*, (12), leads to an LMI which is the lower left 3×3 block of LMI (23) and is always satisfied as long as there is a feasible solution to (23). In summary, feasibility of the LMI condition (23) ensures that the LMI problem (20) is feasible and based on Theorem 2, the proof of Theorem 3 is complete. ■

Remark 1: The choice of slack variables as in (24) is inspired by the authors' previous work [18]. It is noted that choosing lower number of slack variables, *e.g.*, one or two, would be also possible but leads to more conservative results compared to those proposed here. On the other hand, we found out that stacking more than three slack variables in (24) results in a complicated LMI problem, which adds to the design complexity.

V. State-feedback $\mathscr{H}_\infty$ Control Design for Time-delay LPV Systems

In this section, we employ the performance analysis conditions presented in previous section to develop an LMI-based procedure for the sampled-data control design.

Theorem 4: Consider the time-delay LPV system represented by (2). There exist a sampled-data controller of the form (5) such that the corresponding hybrid closed-loop system is asymptotically stable and satisfies $\|z\|_{\mathscr{L}_2} \le \gamma \|w\|_{\mathscr{L}_2}$ for all $\tau(t) \le h$ if there exist a continuously differentiable matrix function $\tilde{P} : \mathbb{R}^s \to \mathbb{S}_+^{n\times n}$, a matrix function $\tilde{K} : \mathbb{R}^s \to \mathbb{S}_+^{n_u\times n}$, constant matrices $\tilde{R}$ and $U \in \mathbb{S}_+^{n\times n}$, a positive scalar γ and real scalars λ_2 and λ_3 such that the LMI condition

$$\begin{bmatrix} -2U & \tilde{P}-\lambda_2 U + AU & -\lambda_3 U + B_2\tilde{K} & B_1 & 0 & h\tilde{R} \\ \star & \dot{\tilde{P}} - \tilde{R} + \lambda_2(AU+UA^T) & \tilde{R}+\lambda_3 UA^T + \lambda_2 B_2\tilde{K} & \lambda_2 B_1 & UC_1^T & 0 \\ \star & \star & -\tilde{R}+\lambda_3(B_2\tilde{K}+\tilde{K}^T B_2^T) & \lambda_3 B_1 & \tilde{K}^T D_{12}^T & 0 \\ \star & \star & \star & -\gamma I & D_{11}^T & 0 \\ \star & \star & \star & \star & -\gamma I & 0 \\ \star & \star & \star & \star & \star & -\tilde{R} \end{bmatrix} < 0 \tag{25}$$

is feasible for any admissible trajectory $\rho(t) \in \mathscr{F}_{\mathscr{P}}^{\nu}$. Then, the corresponding controller gain in (5) is obtained by $K(\rho(t))|_{t=t_k} = \tilde{K}(\rho(t))|_{t=t_k} U^{-1}$

Proof: We first start from the LMI (23) and substitute the system matrices from (8). Then, we place a constraint on the slack variables as $V_1 = V$, $V_2 = \lambda_2 V$ and $V_3 = \lambda_3 V$, where $V \in \mathbb{S}_+^{n\times n}$ and λ_2 and λ_3 are real scalars. Applying the congruent transformation $T = diag(U,U,U,I,I,U)$ with $U = V^{-1}$ and considering $U^T PU = \tilde{P}$, $U^T RU = \tilde{R}$ and $KU = \tilde{K}$, the LMI condition in (25) is obtained and this completes the proof. ■

Remark 2: In the matrix inequality (25), the (2,2) entry includes a derivative term that can be replaced by $\dot{\tilde{P}} = \frac{\partial \tilde{P}}{\partial \rho}\dot{\rho}$. Due to the affine dependency of this matrix inequality on $\dot{\rho}$, it is only required to solve this feasibility problem at vertices of $\dot{\rho}$. Therefore, in matrix inequality (25), one can replace the term $\dot{\tilde{P}}$ with $\sum_{i=1}^{s} \pm \left(\nu_i \frac{\partial \tilde{P}}{\partial \rho}\right)$ [15]. The summation means that every combination of + and - should be included in the inequality. That is, the inequality (25) actually represents 2^s different combinations in the summation.

Remark 3: It is noted that the matrix function $\tilde{K}$ in LMI (25) can be considered to be dependent on the delayed version of scheduling parameter vector, *i.e.*, $\tilde{K}(\rho(t-\tau)) = \tilde{K}(\rho(t))|_{t=t_k}$ for $t_k \le t < t_{k+1}$. This implies that $\tilde{K}$ is updated once at each interval. With this choice of dependency of the control matrix gain on LPV parameter vector, we can replace $\rho(t-\tau(t))$ by $r(t)$ in (25) and treat it as a new parameter vector $r \in \mathscr{F}_{\mathscr{P}}^{\nu}$. In this case, the feasibility or optimization problem corresponding to the sampled-data control can be solved over the new parameter space $\mathscr{F}_{\mathscr{P}}^{\nu} \times \mathscr{F}_{\mathscr{P}}^{\nu}$.

VI. Simulation Results

In this section, we demonstrate the viability of the proposed sampled-data control design method by an example. We consider the following linear time-varying system

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 2\sin(0.2t) & 1.1+\sin(0.2t) \\ -2.2+\sin(0.2t) & -3.3+0.1\sin(0.2t) \end{bmatrix} x(t) + \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix} w(t) \\ &+ \begin{bmatrix} 2\sin(0.2t) \\ 0.1+\sin(0.2t) \end{bmatrix} u(t) \\ z(t) &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t). \end{aligned} \tag{26}$$

A sampled-data controller is designed to stabilize the plant and attenuate the effect of disturbance $w(t)$ on the state $x_2(t)$ while maintaining a reasonable control action. We are also interested in determining the maximum sampling period such that the system remains stable. It is assumed that the system is affected by a rectangular disturbance $w(t)$ in the time interval $t \in [0,5]$. In the model shown above, the sine term is assumed to be an LPV parameter, whose functional representation is not known *a priori* but can be measured in real time. Defining $\rho(t) = \sin(0.2t)$, we obtain an LPV state-space representation with the parameter space $\rho \in [-1,1]$ and $\nu = 0.2$. To solve the LMI problem (25), we have to decide on the function variables $\tilde{P}$ and $\tilde{K}$, as well as the scalars λ_2 and λ_3. A widely used choice of the matrix functions is

$$\begin{aligned} \tilde{P}(\rho) &= \tilde{P}_0 + \rho\tilde{P}_1 + \frac{\rho^2}{2}\tilde{P}_2 + \cdots \\ \tilde{K}(\rho) &= \tilde{K}_0 + \rho\tilde{K}_1 + \frac{\rho^2}{2}\tilde{K}_2 + \cdots \end{aligned} \tag{27}$$

The structure of the matrix functions is part of the design and is an *ad hoc* procedure. It is also noted that the optimal value of γ corresponding to a given bound on sampling rate, *i.e.*, h, is quite sensitive to the scalars λ_2 and λ_3. These two scalars can be optimized by performing a 2-dimensional search. To solve the LMI problem corresponding to the sampled-data control design problem, we consider two cases: (i) with parameter-independent matrices, *i.e.*, $\tilde{P}(\rho) = \tilde{P}_0$ and $\tilde{K}(\rho) = \tilde{K}_0$, and (ii) with second-order polynomials in (27). Table I shows the optimized $\mathscr{H}_\infty$ norm of the closed-loop system with respect to different maximum sampling rates and two choices of parameter-dependency for the matrix functions $\tilde{P}(\rho)$ and $\tilde{K}(\rho)$. Shown in Fig. 2 is the response of the system (26) to the disturbance w with a constant sampling rate $h = 0.3$ and using parameter-independent function variables. The dashed line corresponds to the state x_2 of the continuous-time system and the solid line indicates the staircase control action. The obtained $\mathscr{H}_\infty$ performance level is $\gamma = 0.513$. It is noted that the output response would be improved for smaller sampling rates. Finally, we examine the case of variable sam-

TABLE I

$\mathcal{H}_\infty$ NORM FOR DIFFERENT SAMPLING RATES AND BASIS FUNCTIONS

h	parameter-independent functions	2^{nd}-order polynomial
0.1	0.328	0.325
0.2	0.379	0.375
0.3	0.513	0.495
0.4	1.2	0.905
0.5	Infeasible	1.93
0.6	Infeasible	8.04

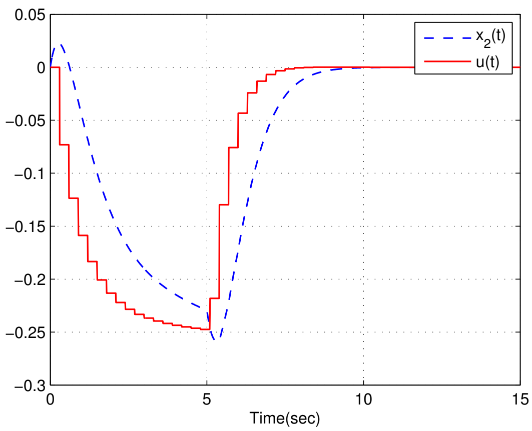

Fig. 2. Output response for $h = 0.3$; the two scalars are tuned as $\lambda_2 = 1.6$ and $\lambda_3 = 0.1$

pling rate, that changes corresponding to the periodic pattern $\{0.3, 0.4, 0.5, 0.3, 0.4, 0.5, \ldots\}^{sec}$. This implies the sampling time instants $\{0, 0.3, 0.7, 1.2, 1.5, 1.9, 2.4, 2.7, \ldots\}^{sec}$. In order to handle the sampling rate of 0.5^{sec}, we use parameter-dependent function variables. In addition, we tune the two scalars for $h = 0.4$ as $\lambda_2 = 1.6$ and $\lambda_3 = 1$ to yield an acceptable performance for various sampling rates around this nominal value. Figure 3 shows the simulation results demonstrating an acceptable disturbance attenuation obtained by the sampled-data controller with a reasonable control action.

VII. Concluding Remarks

In this paper, we employ the so-called input delay approach for sampled-data control design of continuous-time LPV systems by mapping the hybrid closed-loop system to a continuous-time state-delayed LPV system. Then, to ensure the asymptotic stability and $\mathcal{H}_\infty$ performance of the resulting closed-loop hybrid system, we utilize the slack variables to relax the resulting inequality condition in terms of an LMI problem. It is shown that our choice of the Lyapunov-Krasovskii functional results in successfully handling the fast-varying time delays which is essential for sampled-data control design. We further show that the proposed sampled-data control design method can be used for state-feedback design in systems with varying sampling rates.

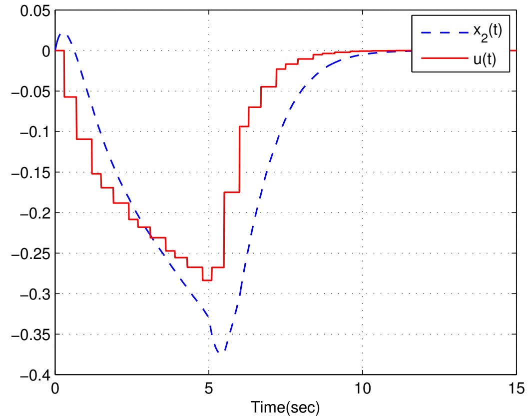

Fig. 3. Output response for variable sampling rate case; the two scalars are tuned as $\lambda_2 = 1.6$ and $\lambda_3 = 1$

References

[1] B.A. Bamieh and J.B. Pearson Jr. A general framework for linear periodic systems with applications to H_∞ sampled-data control. *Automatic Control, IEEE Transactions on*, 37(4):418–435, 1992.

[2] C. Briat. *Robust control and observation of LPV time-delay systems*. PhD thesis, Universitadegli Studi di Siena, 2008.

[3] C. Briat, O. Sename, and J.F. Lafay. Parameter dependent state-feedback control of LPV time-delay systems with time-varying delays using a projection approach. *IFAC World Congress, Seoul*, 2008.

[4] T. Chen and B.A. Francis. *Optimal sampled-data control systems*, volume 124. Springer London, 1995.

[5] E. Fridman. New Lyapunov-Krasovskii functionals for stability of linear retarded and neutral type systems. *Systems & Control Letters*, 43(4):309–319, 2001.

[6] E. Fridman, A. Seuret, and J.P. Richard. Robust sampled-data stabilization of linear systems: an input delay approach. *Automatica*, 40(8):1441–1446, 2004.

[7] E. Fridman, U. Shaked, and V. Suplin. Input/output delay approach to robust sampled-data H_∞ control. *Systems & Control Letters*, 54(3):271–282, 2005.

[8] Micheev Yu. Fridman E. and Sobolev V. Asymptotic analysis of digital control systems. *Automation and remote control*, 49:1175–1180, 1998.

[9] K. Gu, V. Kharitonov, and J. Chen. *Stability of time-delay systems*. Birkhauser, 2003.

[10] J.P. Richard. Time-delay systems: an overview of some recent advances and open problems. *Automatica*, 39(10):1667–1694, 2003.

[11] R.E. Skelton, T. Iwasaki, and K.M. Grigoriadis. *A unified algebraic approach to linear control design*. CRC, 1998.

[12] V. Suplin, E. Fridman, and U. Shaked. Sampled-data H_∞ control and filtering: Nonuniform uncertain sampling. *Automatica*, 43(6):1072–1083, 2007.

[13] K. Tan, K.M. Grigoriadis, and F. Wu. Output-feedback control of LPV sampled-data systems. *International Journal of Control*, 75(4):252–264, 2002.

[14] H.D. Tuan, P. Apkarian, and T.Q. Nguyen. Robust and reduced-order filtering: new LMI-based characterizations and methods. *Signal Processing, IEEE Transactions on*, 49(12):2975–2984, 2001.

[15] F. Wu and K.M. Grigoriadis. LPV systems with parameter-varying time delays: analysis and control. *Automatica*, 37(2):221–229, 2001.

[16] F. Zhang and K.M. Grigoriadis. Delay-dependent stability analysis and H_∞ control for state-delayed LPV system. In *Proc. Mediterrean Conference on Control and Automation*, pages 1532–1537. IEEE, 2005.

[17] X. Zhang, P. Tsiotras, and C. Knospe. Stability analysis of LPV time-delayed systems. *International Journal of Control*, 75(7):538–558, 2002.

[18] R. Zope, J. Mohammadpour, K.M. Grigoriadis, and M. Franchek. Delay-dependent H_∞ control for LPV systems with fast-varying time delays. *Proc. American Control Conference, to appear*, 2012.