2016 American Control Conference (ACC)
Boston Marriott Copley Place
July 6-8, 2016. Boston, MA, USA

Distributed Model Predictive Control of Constrained Spatially-Invariant Interconnected Systems in Input-Output Form*

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Abstract— **This paper proposes a non-iterative, Lyapunov-based distributed model predictive control (MPC) design for invariant spatially-interconnected systems comprised of a network of subsystems with coupled dynamcis and subject to local state and/or input constraints. Considered here is the distributed MPC design for linear systems in input-output form with fixed-structure local state-feedback-like control law. The proposed distributed MPC design approach ensures asymptotic stability and recursive feasibility, and its online implementation can be formulated as solving a linear matrix inequality (LMI) problem defined at the subsystem level. Simulation results using a heat equation in one-dimensional space demonstrate the merits and effectiveness of the proposed approach.**

I. INTRODUCTION

In practice, nearly every application is subject to various design and operational constraints. Model predictive control (MPC) has a long history in industry to deal with system constraints. Not restricted to lumped systems, distributed MPC has become an active research area in the last decade to cope with large, networked systems, which can either be physically coupled, e.g., distributed parameter systems [1], or consist of a network of systems with decoupled dynamics, but coupled via constraints or a common objective, e.g., multi-agent systems [2]. Distributed MPC allows information exchange among a small number of subsystems, as opposed to a centralized MPC scheme. A review of recent results in the design of distributed MPC can be found in [3] and [4].

The vast majority of the published literature, either on control of unconstrained distributed systems or control of constrained systems using anti-windup or MPC approach, rely on state-space representation of plant dynamics, and thus analysis and synthesis conditions are derived in state-space form e.g., [5]. Only very few address these issues based on input-output model, e.g., [6] for unconstrained distributed controller design and [7] for MPC design in lumped systems.

The importance of controller design in input-output form is related to the fact that the experimentally identified models are in input-output form. It is then natural to solve for the controller based on the input-output dynamics and implement it digitally. MPC design based on state-space formulation requires the online measurement or estimation of the states. In practice, an online state estimator is often constructed separately from MPC; it can lead to a non-trivial stability proof. Furthermore, the state-space based controller design normally results in controllers of full order (equal or larger than the plant order), whereas the input-output framework allows to design *fixed-structure* controllers that have predefined orders [8]. The advantage of considering input-output form becomes more obvious when dealing with linear parameter-varying systems [9].

Inspired by the results reported in [6] and [7], this paper studies the distributed MPC design for spatially-interconnected systems in input-output form, where we consider Lyapunov-based stability analysis and a local controller design of fixed structure. We consider the class of spatially-distributed systems, where the attachment of actuators and sensors induces the spatial discretization of the overall system into a network of interconnected subsystems, with each subsystem interacting with its nearest neighbours. Examples include partial differential equation (PDE)-governed distributed systems, environmental systems [10], and canal systems [11] among many others. The distributed MPC should preserve the localized structure of the plant, with stability and recursive feasibility guaranteed.

This paper is organized as follows: Section II reviews the input-output representation for the class of spatially-interconnected systems considered in this paper. In Section III, a novel distributed MPC design approach in input-output form is presented. A numerical example is used in Section IV to illustrate its closed-loop performance. Finally, conclusions are drawn in Section V.

II. PRELIMINARIES

In this paper, we consider the class of spatially-interconnected systems whose subsystems exhibit linear time- and space-invariant (LTSI) properties. This section reviews the two-dimensional difference equation that defines the distributed dynamics at a single subsystem, and presents an implicit representation of the closed-loop system.

A. Input-Output Representation

An LTSI system of one spatial dimension consists of N_s identical subsystems, the dynamics of each governed by a two-dimensional transfer function

$$G(q_t, q_s): \quad A(q_t, q_s)y(k, s) = B(q_t, q_s)u(k, s), \qquad (1)$$

($s = 1, \ldots, N_s$), where q_t and q_s are forward temporal and spatial shift operators, respectively, $y(k, s) \in \mathbb{R}^{n_y}$ and

*This work was supported by the German Research Foundation (DFG) through the research fellowship Li 2763/1-1.

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978-1-4673-8682-1/$31.00 ©2016 AACC

$u(k,s) \in \mathbb{R}^{n_u}$ are the output and input of subsystem s at time instant k, respectively. The polynomials A and B are defined as

$$A(q_t, q_s) = 1 + \sum_{(i_k, i_s) \in M_y} a_{(i_k,i_s)} q_t^{-i_k} q_s^{-i_s}, \tag{2a}$$

$$B(q_t, q_s) = \sum_{(j_k, j_s) \in M_u} b_{(j_k,j_s)} q_t^{-j_k} q_s^{-j_s}, \tag{2b}$$

where M_u and M_y are input and output masks, respectively, determining how temporally- and spatially-shifted inputs and outputs contribute to the dynamics of subsystem s, $a_{(i_k,i_s)}$ and $b_{(j_k,j_s)}$ are coefficients weighting the contribution of time and space samples. An example of masks is shown in Fig. 1, where black dots indicate contributing input and output samples. This choice of masks indicates that output of subsystem s is directly determined by its past input and outputs of itself and neighbouring subsystems $s-1$ and $s+1$.

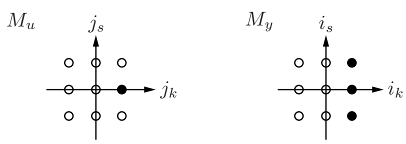

Fig. 1. Input and output masks as an example.

In this paper, we assume that all subsystems interact with neighbouring subsystems in the same pattern, i.e., the input and output masks, as well as the corresponding coefficients, are identical for all subsystems.

Definition 1 (Neighbourhood $\mathcal{N}_i$) *Subsystem j exchanges information with subsystem i ($i \neq j$) if $j \in \mathcal{N}_i$, where $\mathcal{N}_i$ denotes the directly interconnected neighbouring subsystems of subsystem i.*

A system with masks defined as Fig. 1 has a neighbourhood $\mathcal{N}_s = \{s-1, s+1\}$ of size $n_{\mathcal{N}_s} = 2$. Here we focus on undirected communication between subsystems, i.e., $i \in \mathcal{N}_j$ and $j \in \mathcal{N}_i$.

In this paper, we consider the control problem of tracking reference signal $w(k,s)$. The closed-loop system at subsystem s with performance channels incorporated is shown in Fig. 2, where $W_s(q_t,q_s)$ and $W_k(q_t,q_s)$ are weighting filters for shaping the sensitivity and controller sensitivity of the closed-loop system, respectively, $z(k,s) = [z_s^{\mathrm{T}}(k,s), z_k^{\mathrm{T}}(k,s)]^{\mathrm{T}}$ is the corresponding fictitious performance output.

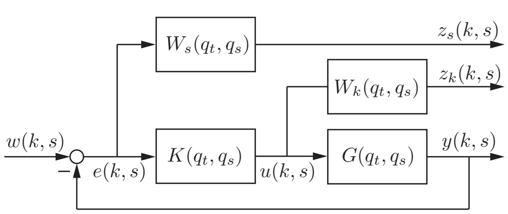

Fig. 2. Generalized plant for subsystem s.

Fixed-structure controller design (in the sense of fixed temporal and spatial orders) allows us to determine the orders of the controller and weighting filters independently via the choices of the input and output masks (M_u^K, M_y^K), ($M_u^{W_s}$, $M_y^{W_s}$) and ($M_u^{W_k}$, $M_y^{W_k}$), respectively. Considered here are strictly proper plant models, whereas the controller and weighting filters can in general be either bi-proper or strictly proper.

It is desired that the controller inherits the spatial structure of the plant, i.e., a distributed controller of an LTSI plant is assumed to be an LTSI system as well. A distributed controller takes the form

$$K(q_t, q_s): \quad A_K(q_t,q_s) u(k,s) = B_K(q_t,q_s) e(k,s). \tag{3}$$

The transfer functions of weighting filters $W_s(q_t,q_s)$ and $W_k(q_t,q_s)$ can be constructed in a similar way.

B. Implicit Representation

Without loss of generality, we consider single-input and single-output subsystem $G(q_t,q_s)$. Let $\bar{A}$ and $\bar{B}$ denote vectors containing polynomial coefficients of $A(q_t,q_s)$ and $B(q_t,q_s)$, respectively, $\xi^y(k,s) \in \mathbb{R}^{n_{\xi^y}}$ and $\xi^u(k,s) \in \mathbb{R}^{n_{\xi^u}}$ the conformable temporally- and spatially-shifted outputs and inputs, respectively. With the coefficient and signal vectors for operators K, W_s and W_k defined accordingly, an implicit representation of the closed-loop system in Fig. 2 can be formulated as

$$\begin{bmatrix} \bar{A} & -\bar{B} & 0 & 0 & 0 \\ \bar{B}_K & \bar{A}_K & 0 & 0 & -\bar{B}_K \\ \bar{B}_{W_s} & 0 & \bar{A}_{W_s} & 0 & -\bar{B}_{W_s} \\ 0 & \bar{B}_{W_k} & 0 & -\bar{A}_{W_k} & 0 \end{bmatrix} \begin{bmatrix} \xi^y(k,s) \\ \xi^u(k,s) \\ \xi^z(k,s) \\ \xi^w(k,s) \end{bmatrix} := H\xi(k,s) = 0, \tag{4}$$

with $\xi \in \mathbb{R}^{n_\xi}$. Note that to ensure the size compatibility of matrix H in case of different operator orders, the missing polynomial terms are filled with zero coefficients; this implies that $n_{\xi^y} = n_{\xi^u} = n_{\xi^z} = n_{\xi^w}$ [6].

In (4), each signal vector $\xi^i(k,s)$ ($i = y,u,z,w$) can be partitioned into three parts as

$$\xi^i(k,s) = \begin{bmatrix} \xi_t^{i\mathrm{T}}(k,s) & \xi_+^{i\mathrm{T}}(k,s) & \xi_-^{i\mathrm{T}}(k,s) \end{bmatrix}^{\mathrm{T}}, \tag{5}$$

where $\xi_t^i(k,s)$ denotes the temporal variables, $\xi_+^i(k,s)$ and $\xi_-^i(k,s)$ the positive and negative spatial variables, respectively. Subsystems exchange information through spatial states. Furthermore, a state-like vector $x(k,s) \in \mathbb{R}^{n_x}$ can be extracted from $\xi(k,s)$ as

$$x(k,s) = \begin{bmatrix} x_t(k,s) \\ x_+(k,s) \\ x_-(k,s) \end{bmatrix}, \quad \mathbf{\Delta} x(k,s) = \begin{bmatrix} x_t(k+1,s) \\ x_+(k,s+1) \\ x_-(k,s-1) \end{bmatrix},$$

where the augmented shift operator $\mathbf{\Delta}$ is defined as $\mathbf{\Delta} = \mathrm{diag}\{q_t, q_s, q_s^{-1}\}$. Permutation matrices $\Pi_1 = \mathrm{diag}\{\Pi_{1t}, \Pi_{1+}, \Pi_{1-}\}$ and $\Pi_2 = \mathrm{diag}\{\Pi_{2t}, \Pi_{2+}, \Pi_{2-}\}$ are properly selected such that

$$x(k,s) = \Pi_1 \xi(k,s), \quad \mathbf{\Delta} x(k,s) = \Pi_2 \xi(k,s).$$

Denote the closed-loop system by L. A spatially-interconnected system in one-dimensional space is shown in Fig. 3.

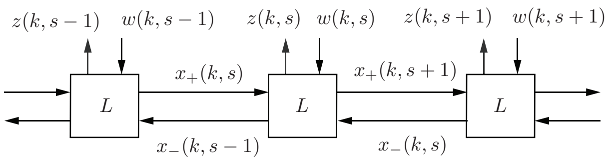

Fig. 3. Spatially-interconnected system in one-dimensional space.

III. Distributed MPC Design

When dealing with a spatially-interconnected system, a centralized MPC scheme requires that each subsystem exchanges information with all other subsystems—it renders both the computation and implementation intractable. Although a distributed MPC approach is often more conservative than the central one, by exploiting the localized nature of subsystems, a distributed MPC based on information received from neighbouring subsystems is of smaller order and much easier to handle.

The following assumptions are made for the distributed MPC design:

i) There are no disturbance and noise present in the closed-loop system and no modelling error of the plant.

ii) For the sake of presentation simplicity, only input constraints are considered here. Furthermore, we assume input constraints imposed at all subsystems are identical, i.e., $\mathbb{U}^1 = \mathbb{U}^2 = \ldots = \mathbb{U}^{N_s}$, with the compact set $\mathbb{U}^s$ $(s = 1, \ldots, N_s)$ defined as

$$\mathbb{U}^s = \{u(k,s) \in \mathbb{R} | -u_{\max} \leq u(k,s) \leq u_{\max}\}, \quad (6)$$

where $u_{\max}$ is the maximum capacity of a physical actuator.

The global control constraint is thus denoted as the Cartesian product of all subsystem constraints

$$\mathbb{U} = \mathbb{U}^1 \times \mathbb{U}^2 \times \ldots \mathbb{U}^{N_s}. \quad (7)$$

Although only the input constraints are addressed, the approach presented in this paper can be easily extended to more general cases accounting for state or combined constraints.

Taking the closed-loop system in Fig. 2 into account, we propose a cost function at subsystem s as

$$V_N^s(x,u,w) = \sum_{i=0}^{N-1} l(\tilde{e}(k+i+1,s), u(k+i,s)) + V_f(x_t(k+N+1,s)), \quad (8)$$

where N is the prediction horizon, control inputs $u(k+i,s)$ are the decision variables, $\tilde{e}(k+i+1,s) \in \mathbb{R}^{n_{\mathcal{N}_s}+1}$ is a vector formed by stacking up the error signals $e(k+i+1,j)$, $j \in \mathcal{N}_s \cup s$. The functions $l(\cdot)$ and $V_f(\cdot)$ are the so-called stage cost and terminal cost, respectively. We define the stage cost function as a positive definite function

$$l(\cdot) = \sum_{j \in \mathcal{N}_s \cup s} q_j e^2(k+i+1,j) + r u^2(k+i,s),$$

with weighting factors $q_j > 0$ and $r > 0$. The local stage cost function takes the deviations between outputs and references at subsystem s, as well as its neighbourhood $\mathcal{N}_s$, as performance measure to account for the influence of the input trajectory $u(k+i,s)$ to s and $\mathcal{N}_s$. Thus a single subsystem does not compete for its own minimum cost at the price of an increased cost at other subsystems. A collective improvement can be expected through cooperation among localized subsystems.

The distributed MPC problem for a spatially-interconnected system in the presence of input constraint can be formulated at subsystem s as

$$\mathbb{P}^s: \quad \min_u V_N^s(x,u,w) \quad (9a)$$

$$\text{s. t.} \quad H\xi(k,s) = 0, \quad (9b)$$

$$u(k+i,s) \in \mathbb{U}^s, \quad \forall i = 0,1,\ldots,N-1 \quad (9c)$$

$$x_t(k+N+1,s) \in \mathbb{X}_f^s, \quad (9d)$$

where $\mathbb{X}_f^s$ is the terminal set containing the origin, whose significance will become clear soon. Similar to the input constraint, the global terminal set can be defined as the Cartesian product of subsystem ones

$$\mathbb{X}_f = \mathbb{X}_f^1 \times \mathbb{X}_f^2 \times \ldots \mathbb{X}_f^{N_s}. \quad (10)$$

To ensure stability and recursive feasibility when solving finite horizon optimization problem, the selection of the terminal cost $V_f(\cdot)$, the terminal set $\mathbb{X}_f^s$, and a local control law valid in $\mathbb{X}_f^s$ plays an important role. In the rest of the section, the offline computation of a local control law and the terminal set, and the formulation of the online optimization problem $\mathbb{P}^s$ (9) in terms of linear matrix inequalities (LMIs) are discussed.

A. Local Control Law and Terminal Cost

For a constrained system, a global Lyapunov function can hardly be found which fulfils Lyapunov-based stability conditions [12]. Instead, a dual mode control is often employed: within the prediction horizon, the predicted inputs are optimization variables; over the remaining infinite horizon of mode 2, inputs are determined by a stabilizing control law. The objective of the MPC design is to steer the system state into the terminal set at the end of the prediction horizon, where a local control law takes over. The computation of the terminal set will be address in Section III-B. In this section, analysis and synthesis conditions for a stabilizing control law are provided.

Consider a Lyapunov function candidate as

$$V(x_t(k,s)) = x_t(k,s)^{\mathrm{T}} P_t x_t(k,s), \quad P_t > 0. \quad (11)$$

We are interested in controller design which establishes asymptotic stability, as well as certain performance criterion of the closed-loop system. Take the induced $\mathcal{L}_2$ norm as performance measure. The closed-loop system is said to have a quadratic performance γ if

$$\sum_{s=1}^{N_s} \sum_{k=0}^{N_t} [*]^{\mathrm{T}} \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix} \begin{bmatrix} z(k,s) \\ w(k,s) \end{bmatrix} < 0. \quad (12)$$

Analysis conditions for the closed-loop system inside the terminal set, i.e., $H\xi = 0$, $x_t(k,s) \in \mathbb{X}_f^s$, can be derived by extending results for unconstrained LTSI systems reported in [6], where distributed controller design has the size of a single subsystem.

Theorem 1 *Assuming the well-posedness of the closed-loop system (4), it is asymptotically stable and has a quadratic performance γ ($\gamma > 0$) inside the terminal set, if there exist a matrix F and a symmetric matrix P, such that*

$$P = \begin{bmatrix} P_t & \\ & P_s \end{bmatrix}, \quad P_t > 0, \quad \det(\mathrm{P_s}) \neq 0, \tag{13a}$$

$$U(P) + M(\gamma) + FH + H^{\mathrm{T}}F^{\mathrm{T}} < 0, \tag{13b}$$

with

$$U(P) = \Pi_4^{\mathrm{T}} P \Pi_4 - \Pi_3^{\mathrm{T}} P \Pi_3 + \Pi_5^{\mathrm{T}} S \Pi_5,$$
$$M(\gamma) = \mathrm{diag}\{\mathbf{0}, \mathrm{I}, -\gamma^2 \mathrm{I}\},$$

and

$$\Pi_3 = \begin{bmatrix} \Pi_{1t}^{\mathrm{T}} & \Pi_{1+}^{\mathrm{T}} & \Pi_{2-}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}, \quad \Pi_4 = \begin{bmatrix} \Pi_{2t}^{\mathrm{T}} & \Pi_{2+}^{\mathrm{T}} & \Pi_{1-}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}.$$

Proof: It has been proven in [6] that an unconstrained LTSI system is asymptotic stable with

$$V(x_t(k+1,s)) - V(x_t(k,s)) < 0 \tag{14}$$

fulfilled and has quadratic performance γ (12), if there exist P and F such that (13a) and

$$\xi^{\mathrm{T}}(\Pi_4^{\mathrm{T}} P \Pi_4 - \Pi_3^{\mathrm{T}} P \Pi_3 + M(\gamma) + FH + H^{\mathrm{T}} F^{\mathrm{T}})\xi < 0. \tag{15}$$

To include the stage cost $l(\cdot)$ in (14) [12] such that

$$V(x_t(k+1,s)) - V(x_t(k,s)) \leq -l(\tilde{e}(k-1,s), u(k-1,s)),$$

a permutation matrix Π_5 is introduced with

$$\Pi_5 \xi(k,s) = \begin{bmatrix} \tilde{e}(k-1,s) \\ u(k-1,s) \end{bmatrix}. \tag{16}$$

A weighting matrix $S = \mathrm{diag}\{\dots, q_j, \dots, r\}, \quad j \in \mathcal{N}_s \cup s$ with predetermined weighting factors is constructed in a conformable way. The stage cost function can then be written in a quadratic form as

$$l(\tilde{e}(k-1,s), u(k-1,s)) = \xi^{\mathrm{T}} \Pi_5^{\mathrm{T}} S \Pi_5 \xi. \tag{17}$$

The application of the S-procedure [13] to (15) and (17) results in (13b). ∎

Provided the analysis condition, controller synthesis condition can be accordingly derived by solving

$$\min_{\bar{A}_K, \bar{B}_K, F, P} \gamma, \tag{18}$$

such that (13a) and (13b) hold.

Note that for controller synthesis, (13b) turns into a bilinear matrix inequality (BMI) condition due to the fact that the matrix H is a function of the decision variables $\bar{A}_K$ and $\bar{B}_K$. A DK-iteration based approach has been demonstrated in [6] to solve (13) effectively.

After the temporal Lyapunov matrix P_t and a controller in the form of (3) have been obtained, the terminal cost $V_f(\cdot)$ in (8) is often chosen to be the Lyapunov function [12], i.e.,

$$V_f(x_t(k,s)) = x_t^{\mathrm{T}}(k,s) P_t x_t(k,s). \tag{19}$$

Note that the controller can be rewritten in a state-feedback-like control law

$$u(k,s) = K_f^s \xi(k,s), \tag{20}$$

where the feedback gain $K_f^s \in \mathbb{R}^{1 \times n_\xi}$ is a function of $\bar{A}_K$ and $\bar{B}_K$.

Remarks:

- The spatial state vectors $x_+(k,s)$ and $x_-(k,s)$ and the spatial Lyapunov matrix P_s are neither included in the Lyapunov function, nor in the terminal cost. This is due to the spatial non-causality: the spatial dynamics cancel out after summed up over spatial coordinate, see [14] [6] for more details. Furthermore, the indefiniteness of the spatial Lyapunov matrix P_s violates the positive definiteness of a Lyapunov function.
- Energy dissipation of the global system implies stability. Conservatism may be induced when energy decreased monotonously at each subsystem is required. Possible solutions for relaxing this constraint can be found in [15].

B. Terminal Set

Stability requires that the overall terminal set $\mathbb{X}_f$ (10) is control invariant [12] for the controlled system under constraints. In this work, by allowing certain conservatism, we ensure the control invariance of the subsystem terminal set $\mathbb{X}_f^s$, i.e., the existence of K_f^s for all $x_t(k,s) \in \mathbb{X}_f^s$, such that $K_f^s \xi(k,s) \in \mathbb{U}^s$, and $x_t(k+1,s) \in \mathbb{X}_f^s$. A common choice for $\mathbb{X}_f^s$ is a suitably small sub-level set of $V_f(x_t(k,s))$, i.e., $\mathbb{X}_f^s$ is an ellipsoidal set [12] defined as

$$\mathbb{X}_f^s := \{x_t(k,s) | V_f(x_t(k,s)) \leq \alpha^s\}, \quad \alpha^s > 0. \tag{21}$$

The terminal sets $\mathbb{X}_f^s$ can be computed offline by maximizing α^s, such that the input constraints are satisfied inside the terminal region, i.e., $K_f^s \xi(k,s) \in \mathbb{U}^s$, $\forall x_t(k,s) \in \mathbb{X}_f^s$.

The computation of $\mathbb{X}_f^s$ is formulated as

$$\max_{\alpha^s, \xi} \quad \alpha^s \tag{22a}$$
$$\text{s. t.} \quad x_t^{\mathrm{T}}(k,s) P_t x_t(k,s) \leq \alpha^s, \tag{22b}$$
$$|K_f^s \xi(k,s)| \leq u_{\max}. \tag{22c}$$

Inequality (22b) can be rewritten as a function of $\xi(k,s)$ as

$$\xi^{\mathrm{T}}(k,s) \Pi_{1t}^{\mathrm{T}} P_t \Pi_{1t} \xi(k,s) \leq \alpha^s. \tag{23}$$

By eliminating $\xi(k,s)$ in (22c) and (23), (22) can be reformulated as

$$\max_{\tilde{\alpha}^s} \quad \tilde{\alpha}^s \tag{24a}$$
$$\text{s. t.} \quad \tilde{\alpha}^s K_f^s (\Pi_{1t}^{\mathrm{T}} P_t \Pi_{1t})^{-1} K_f^{s\mathrm{T}} \leq u_{\max}^2, \tag{24b}$$

with $\tilde{\alpha}^s = (\alpha^s)^2$. After the maximal $\tilde{\alpha}^s$ (or α^s) has been computed, the local terminal set $\mathbb{X}_f^s$ is determined.

Remarks:

- Recursive feasibility has been guaranteed due to the control invariance of the terminal set. Once the terminal state enters the terminal set, the stabilizing local control law ensures that the closed-loop state trajectory stays inside, and drives the state converging either to the origin (in regulator problem) or a steady state (reference tracking).
- For reference tracking, the steady-state set $\bar{\mathbb{X}}^s$ needs to be a subset of the terminal set $\bar{\mathbb{X}}^s \subset \mathbb{X}_f^s$ to ensure feasibility, i.e., $\bar{\mathbb{X}}^s = \{\bar{x}_t(k,s) | \bar{x}_t^{\mathrm{T}}(k,s) P_t \bar{x}_t(k,s) < \alpha^s\}$, where $\bar{x}_t(k,s)$ denotes the temporal state vector in steady state. If this condition is violated, the desired steady states are unreachable. To cope with it, the concept of pseudo setpoints has been employed in [16] to handle infeasible references.

Provided the previous discussions on the offline computation of the local control law, the terminal cost, and the terminal set, stability and performance conditions inside the terminal set can be summarized as follows.

Theorem 2 *An LTSI system whose subsystem dynamics are defined by (4) under constraint $u(k,s) \in \mathbb{U}^s$ $(s = 1, \ldots, N_s)$ is asymptotically stable and has a quadratic performance γ inside the terminal set (21), if there exist a local Lyapunov function (11) and a local control law (20) satisfying (13) for all $x_t(k,s) \in \mathbb{X}_f^s$.*

C. Online Optimization Problem

After the offline computation of a local control law $K_f^s(\cdot)$, the terminal cost $V_f(\cdot)$ and local terminal set α^s, the optimization problem (9) is online implemented simultaneously at all subsystems at each time instant. In this section, the online convex optimization problem (9) is to be characterized in terms of LMIs.

The minimization of the cost function $V_N^s(x,u,w)$ equalizes the minimization of a positive scalar β, such that $V_N^s(x,u,w) < \beta$. By rewriting (8) in a quadratic form, with the application of Schur complement, the minimization problem (9a) can then be formulated as an LMI condition

$$\min_{\beta,u} \quad \beta, \tag{25}$$

$$\text{s.t.} \begin{bmatrix} Q^{-1} & & & \tilde{e}_p(k,s) \\ & R^{-1} & & u_p(k,s) \\ & & P_t^{-1} & x_t(k+N+1,s) \\ *^{\mathrm{T}} & *^{\mathrm{T}} & *^{\mathrm{T}} & \beta \end{bmatrix} > 0, \tag{26}$$

where

$$\tilde{e}_p(k,s) := \begin{bmatrix} \tilde{e}(k+1,s) \\ \vdots \\ \tilde{e}(k+N-1,s) \end{bmatrix}, u_p(k,s) := \begin{bmatrix} u(k,s) \\ \vdots \\ u(k+N-1,s) \end{bmatrix},$$

$$x_t(k+N+1,s) = [y(k+N,s), \quad u(k+N,s), \quad z(k+N,s), \quad w(k+N,s)]^T,$$

Q and R are diagonal matrices containing q_j $(j \in \mathcal{N}_s \cup s)$ and r, respectively, P_t has been solved from (13).

Given the definition of the ellipsoidal terminal set (21), the constraints (9c)-(9d) are characterized in LMI form as

$$-u_{\max} \le u(k+i,s) \le u_{\max}, \quad i = 0, \ldots, N-1, \tag{27}$$

$$\begin{bmatrix} P_t^{-1} & x_t(k+N+1,s) \\ *^{\mathrm{T}} & \alpha^s \end{bmatrix} > 0. \tag{28}$$

The online implementation of the proposed distributed MPC can be summarized as follows:

Algorithm 1 Distributed MPC Design (online)

1: for k = 1 to N_t
2: solve (25)-(28) for $u_p(k,s)$, $s = 1, \ldots, N_s$, in parallel at all subsystems
3: apply $u(k,s)$ as control action to subsystem s
4: end

Remarks:

- The deviation $\tilde{e}$ in (26) involves the computation of predicted outputs at subsystems s and $\mathcal{N}_s$. The decision variables $u_p(k,s)$ at subsystem s and spatial information received from its neighbourhood $\mathcal{N}_s$ contribute to the predicted output $y(k+i,s)$, while the updated $y(k+i,s)$ in turn influences the predicted outputs $y(k+i+1,j)$, $j \in \mathcal{N}_s$, due to the fact that we consider undirected communication between subsystems. To keep the computation at the subsystem level, during the implementation of (9) at subsystem s, we assume that information that the neighbourhood $\mathcal{N}_s$ receives from their own neighbourhoods remains the same as the last time instant before the new optimization starts, except the updated $y(k+i,s)$.
- As the decision variables, a trajectory of inputs $u_p(k,s)$ is to be optimized. Nevertheless, the computation of the terminal state $x_t(k+N+1,s)$ in (26) and (28) requires the value of input at time $k+N$ (one step after the prediction horizon), which can be computed using the local control law valid in the terminal set as (20).
- The shaping filters W_s and W_k can be selected either as temporal or spatio-temporal systems. The predicted fictitious outputs $z(k+N,s)$ in (26) and (28) can be computed in the same way as the predicted output $y(k+N,s)$ using their governing transfer functions.

IV. Simulation Results

To demonstrate the performance of the proposed approach, the example used in [6] – heat equation that describes the distribution of heat in a region – is borrowed here. Its governing PDE is given by

$$\frac{\partial y(k,s)}{\partial t} - \kappa \frac{\partial^2 y(k,s)}{\partial x^2} = u(t,s), \tag{29}$$

where the positive constant κ denotes the thermal diffusivity.

Let the sampling time and sampling space be chosen as $T_t = 0.01$ s and $T_s = 0.25$ m, respectively, and $\kappa = 1$. After applying the finite difference method to discretize (29) both in time and space with $N_s = 21$, the resulting two-dimensional difference equation has the input and output masks as shown in Fig. 1, and the vectors of polynomial coefficients

$$\bar{A} = \begin{bmatrix}1 & -1+2\nu & -\nu & -\nu\end{bmatrix}, \quad \bar{B} = \begin{bmatrix}0 & T_t & 0 & 0\end{bmatrix},$$

with $\nu = T_t/T_s^2$.

The structure of the local controller law is fixed as

$$\bar{A}_K = \begin{bmatrix}1\ a^K_{(1,0)}\ a^K_{(1,-1)}\ a^K_{(1,1)}\end{bmatrix},$$
$$\bar{B}_K = \begin{bmatrix}b^K_{(0,0)}\ b^K_{(1,0)}\ b^K_{(1,-1)}\ b^K_{(1,1)}\end{bmatrix}.$$

A generalized plant shaping the sensitivity of the closed-loop system is considered, with the weighting filter W_s chosen as a temporal system

$$\bar{A}_{W_s} = \begin{bmatrix}1 & -0.999 & 0 & 0\end{bmatrix}, \quad \bar{B}_{W_s} = \begin{bmatrix}0 & 0.02 & 0 & 0\end{bmatrix}.$$

The choices of the controller and the weighting filter lead to the vector $\xi \in \mathbb{R}^{16}$, with the extracted temporal state vector

$$x_t(k,s) = \begin{bmatrix}y(k-1,s)\ u(k-1,s)\ z(k-1,s)\ w(k,s-1)\end{bmatrix}^{\mathrm{T}}.$$

An input constraint $u_{\max} = 1$ is imposed on all subsystems. The local control law and Lyapunov function are computed by solving (13). The performance of the proposed distributed MPC approach is evaluated by tracking a unit step reference at subsystem 11 from time instant 10 to 30, i.e., $w(10:30, 11) = 1$. The steady-state set with respect to the given reference is computed by $\bar{\alpha}^s = \bar{x}_t^{\mathrm{T}}(k,s)P_t\bar{x}_t(k,s)$. That $\bar{\alpha}^s < \alpha^s$ with α^s computed from (24) implies the feasibility of tracking the given reference.

The prediction horizon is chosen as $N = 5$. Fig. 4 shows the comparisons of closed-loop response and control input at subsystem 11. Both the blue and red curves are generated by implementing the control law derived in Section III-A without and with input constraint imposed, respectively. After imposing the input constraints (realized by connecting a saturation block between the controller and the plant), the rise time (red) increases due to the slightly saturated control input. The distributed MPC controller (black), taking the constraints into account, guarantees stability of the controlled system and achieves an effective reference tracking, without the imposed constraints violated.

V. Conclusion

In this paper, we considered a distributed MPC design for spatially-interconnected systems whose dynamics are governed by a two-dimensional difference equation at subsystems and can be represented in an implicit input-output form. We showed the computation of the local control law, terminal cost and terminal set to ensure Lyapunov-based stability and recursive feasibility. The proposed MPC design conforms to the localized property of subsystems, that information exchange takes place only between neighbouring subsystems. Closed-loop performance was demonstrated using a numerical example.

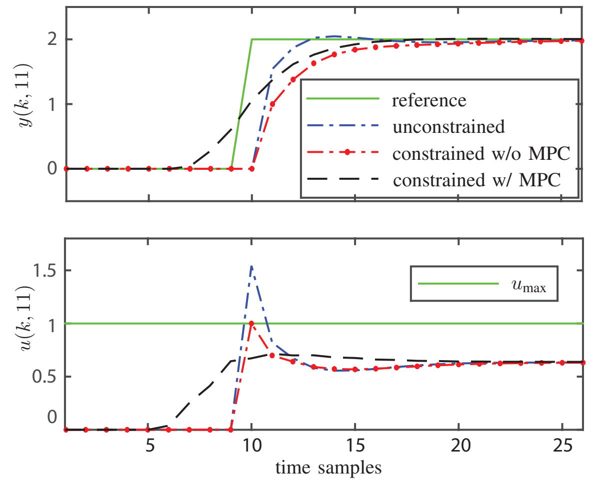

Fig. 4. Comparisons of the closed-loop response (upper) and control input (lower) at subsystem 11: unconstrained closed-loop system (blue), and constrained systems without (red) and with (black) distributed MPC.

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