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A New Method of Parameter-varying Filter Design for LPV Systems with Time-varying State Delay

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***Abstract*— A filter design method for continuous-time linear parameter-varying (LPV) systems with time-varying delay is presented in this paper. The design objective is to ensure that the estimation error system is stable and a prescribed level of performance, namely, the induced energy-to-energy gain from the disturbance input to the estimation error signal, is satisfied. The filter synthesis conditions are formulated in terms of linear matrix inequality (LMI) optimization problems. The proposed design method is finally validated through a numerical example and the results are compared with a related study.**

I. INTRODUCTION

In estimation theory, filters utilize the output measurements from a dynamic system to estimate the outputs of interest, e.g., a combination of the system states. From this point of view, an observer which estimates the nonmeasurable states of a system can be regarded as a filter. A filter can also be used to detect faults in dynamic systems by comparing selected outputs of the physical system with those produced by the filter, which represent the healthy system output [1], [3]. The performance of a filter is often assessed using the output estimation error. To this aim, especially when the statistical information is unknown, the $\mathcal{H}_\infty$ filtering method can be employed to minimize the energy of the estimation error signal for the worst bounded energy disturbance input [7], [11].

In the present paper, we propose a method for filter design in linear parameter-varying (LPV) systems with time-varying state delay. It is noted that a system with delay in the input can be readily converted to one with state delay. In addition to the time-varying nature of the LPV systems, the existence of delay can add to the complexity of the aforementioned filter design problem. The literature on stability analysis, filtering and control of time-delay systems has become rich in the past few years (see, *e.g.*, among many other references, [4], [10] for LTI and [5], [6], [15] for LPV time-delay systems). The existing criteria for analysis of time delay systems are categorized into either delay-independent or delay-dependent approaches. In the delay-independent approach, a controller is designed such that the system remains stable regardless of the time delay magnitude. In contrast, by considering the information about the size of the time delay, the delay-dependent approach leads to generally less conservative results specially for smaller time delays.

There is a close connection between the solutions to the $\mathcal{H}_\infty$ filtering problem and the $\mathcal{H}_\infty$ control problem. However, it is rewarding to derive an independent solution for the filtering problem, which results in a simpler set of design conditions. In the last decade, the filtering problem for LPV systems with time delay has been studied but reported mostly for constant time delays [5]. According to the literature, two classes of filters are often utilized: the delay-free filter and the delayed filter (that has a state-delay term in filter dynamics). The latter structure can potentially lead to better performance since the structure of the filter is similar to that of the plant. To the best of our knowledge, the most dependable solution to treat time-varying delays has been offered in [6], satisfying both $\mathcal{H}_2$ and $\mathcal{H}_\infty$ performance measures simultaneously. However, this approach gives linear matrix inequality (LMI) conditions that depend on the rate of the delay rather than the delay itself. One major issue many of the methods in the literature suffer from is that they fail to handle fast varying time delay, i.e., when the rate of the time delay is grater than or equal to unity.

Contribution of the present paper is as follows. We propose a method for the design of filters for LPV systems including time-varying state delay. The design can guarantee asymptotic stability of the error estimation system and a specified level of performance, namely the energy-to-energy gain from external disturbance to the estimation error. The main result of this paper is inspired by the method proposed recently by the authors [17] for continuous-time controller synthesis of state-delayed LPV systems. The key point to address this problem is to find a parameter-dependent Lyapunov-Krasovskii functional that results in a delay-dependent synthesis method to handle fast-varying time delays. To ensure that the solution to the synthesis problem is in an LMI form, we introduce slack variables (see, e.g., [13]) to relax the resulting conditions in terms of an LMI problem. Using the derived formulation based on the slack variables, we then obtain the synthesis conditions for LPV filter design. The authors have formerly completed a study in filtering of LPV systems using the lifting method [9]. They also have utilized similar approach in this paper in the sampled-data control problem of LPV systems with delay [8].

The notation used in this paper is standard. $\mathbb{R}$ denotes the set of real numbers. $\mathbb{R}^n$ and $\mathbb{R}^{k\times m}$ are used to denote the set of real vectors of dimension n and the set of real $k \times m$ matrices, respectively. In addition, $\mathbb{S}^{n\times n}$ and $\mathbb{S}_+^{n\times n}$ denote the set of real symmetric $n \times n$ matrices

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and symmetric positive definite matrices, respectively. In a symmetric matrix, the asterisk $\star$ in the (i, j) element denotes transpose of the (j, i) element.

II. Preliminaries and Problem Statement

We consider the following state-space representation for an LPV system

$$\begin{aligned}\dot{x}(t) &= A(\rho(t))x(t) + A_h(\rho(t))x(t-h(\rho(t))) + B_1(\rho(t))w(t)\\ z(t) &= C_1(\rho(t))x(t) + C_{1h}(\rho(t))x(t-h(\rho(t))) + D_{11}(\rho(t))w(t)\\ y(t) &= C_2(\rho(t))x(t) + C_{2h}(\rho(t))x(t-h(\rho(t))) + D_{21}(\rho(t))w(t)\\ x(\theta) &= \phi(\theta) \qquad \forall t \in [-h_m, 0],\end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $z(t) \in \mathbb{R}^{n_z}$ is the output to be estimated, $y(t) \in \mathbb{R}^{n_y}$ is the measurement vector, $w(t) \in \mathbb{R}^{n_w}$ is the exogenous disturbance vector containing both process and measurement noises with finite energy. The system matrices $A(\cdot)$, $A_h(\cdot)$, $B_1(\cdot)$, $C_1(\cdot)$, $C_{1h}(\cdot)$, $D_{11}(\cdot)$, $C_2(\cdot)$, $C_{2h}(\cdot)$ and $D_{21}(\cdot)$ are real continuous functions of a time-varying parameter vector $\rho(t)$ and are of appropriate dimensions. In the model above, $h(\cdot)$ is a differentiable scalar function denoting the parameter-dependent time delay that satisfies $0 \le h(\cdot) \le h_m$. The initial condition $\phi(\cdot)$ determines the integral solution of (1) uniquely. The parameter vector $\rho(t) \in \mathcal{F}_{\mathcal{P}}^{v}$ is assumed to be measurable in real-time, where $\mathcal{F}_{\mathcal{P}}^{v}$ is the set of allowable parameter trajectories defined as

$$\begin{aligned}\mathcal{F}_{\mathcal{P}}^{v} \equiv \{\rho(t) \in C(\mathbb{R}, \mathbb{R}^s) : \rho(t) \in \mathcal{P}, |\dot{\rho}_i(t)| \le v_i\\ i = 1, 2, ..., s \quad \forall t \in \mathbb{R}_+\},\end{aligned} \tag{2}$$

where $C(\mathbb{R}, \mathbb{R}^s)$ is the set of continuous-time functions from $\mathbb{R}$ to $\mathbb{R}^s$, $\mathcal{P}$ is a compact set of $\mathbb{R}^s$, and $\{v_i\}_{i=1}^{s}$ are nonnegative numbers. The constraints in (2) imply that the parameter trajectories and their variations are bounded. Next, consider an n^{th}-order continuous-time parameter-varying filter F represented by the following state-space model

$$\begin{aligned}\dot{x}_F(t) = A_F(\rho(t))x_F(t) &+ A_{hF}(\rho(t))x_F(t-h(\rho(t)))\\ &+ B_F(\rho(t))y(t)\\ \hat{z}(t) = C_F(\rho(t))x_F(t) &+ C_{hF}(\rho(t))x_F(t-h(\rho(t)))\\ &+ D_F(\rho(t))y(t),\end{aligned} \tag{3}$$

where $x_F(t) \in \mathbb{R}^n$ is the filter state vector and $\hat{z}(t) \in \mathbb{R}^{n_z}$ is the estimate of the output vector $z(t)$ in (1). The filter matrices $A_F(\cdot)$, $A_{hF}(\cdot)$, $B_F(\cdot)$, $C_F(\cdot)$, $C_{hF}(\cdot)$ and $D_F(\cdot)$ are real continuous functions of the parameter vector $\rho(t)$, which are of appropriate dimensions. In this paper, we only consider the full-order filter design problem, where the filter has the same order as the plant. The output estimation is achieved by feeding the information from the measurement signal $y(t)$ and the scheduling parameter $\rho(t)$ to the filter. To this purpose, we define the estimation error as $e(t) = z(t) - \hat{z}(t)$. For the error system that relates the disturbance signal $w(t)$ to the estimation error signal $e(t)$, the induced $\mathcal{L}_2$-gain is defined as

$$\|T_{we}\|_{i,2} = \sup_{\rho \in \mathcal{F}_{\mathcal{P}}^{v}} \sup_{w \in \mathcal{L}_2 - \{0\}} \frac{\|e\|_{\mathcal{L}_2}}{\|w\|_{\mathcal{L}_2}}, \tag{4}$$

where T_{we} is the operator inducing the disturbance w to the estimation error e. This value, also known as energy-to-energy gain of the augmented system, indicates the worst case output energy $\|e\|_{\mathcal{L}_2}$ over all bounded energy disturbances $\|w\|_{\mathcal{L}_2}$ for all admissible parameter vector $\rho(t) \in \mathcal{F}_{\mathcal{P}}^{v}$. In this paper, we aim to design the filter F so that the filtering error system consisting of the plant model (1) and the filter (3) is asymptotically stable, and also the corresponding energy-to-energy gain is minimized, i.e.,

$$\min_{F} \|T_{we}\|_{i,2}. \tag{5}$$

Instead of the optimal design problem (5), one can solve the γ-suboptimal energy-to-energy gain in which a filter F is sought such that

$$\|T_{we}\|_{i,2} < \gamma, \tag{6}$$

where γ is a given positive scalar. If the inequality (6) holds true, then the estimation error energy will be bounded by $\gamma \|w\|_{\mathcal{L}_2}$ for any nonzero disturbance w with bounded energy.

Next, we augment the plant model (1) with the filter (3) to obtain the state-space representation of the error system. Defining

$$\bar{x}(t) = \begin{bmatrix} x(t) \\ x_F(t) \end{bmatrix}, \tag{7}$$

we have

$$\begin{aligned}\dot{\bar{x}}(t) &= \bar{A}\bar{x}(t) + \bar{A}_h\bar{x}(t-h) + \bar{B}w(t)\\ z(t) &= \bar{C}\bar{x}(t) + \bar{C}_h\bar{x}(t-h) + \bar{D}w(t)\end{aligned} \tag{8}$$

where

$$\begin{aligned}\bar{A} &= \begin{bmatrix} A & 0 \\ B_F C_2 & A_F \end{bmatrix}, \bar{A}_h = \begin{bmatrix} A_h & 0 \\ B_F C_{2h} & A_{hF} \end{bmatrix}\\ \bar{B} &= \begin{bmatrix} B_1 \\ B_F D_{21} \end{bmatrix}, \bar{C} = \begin{bmatrix} C_1 - D_F C_2 & -C_F \end{bmatrix}\\ \bar{C}_h &= \begin{bmatrix} C_{1h} - D_F C_{2h} & -C_{hF} \end{bmatrix}, \bar{D} = D_{11} - D_F D_{21}\end{aligned} \tag{9}$$

which is a continuous-time LPV state delay system. For the sake of simplicity, throughout the paper, we may drop the dependency of the matrices on the parameter vector. Next, we provide some useful lemmas that will play a key role in the proofs of the main results of the paper.

Lemma 1: (Cauchy-Schwarz Inequality [16]): For any positive definite matrix P, any $v(\alpha) \in \mathbb{R}^n$ and any positive scalar h, the following inequality conditions always holds true,

$$h \int_{t-h}^{t} v(\alpha)^T P v(\alpha) d\alpha \ge \left[\int_{t-h}^{t} v(\alpha) d\alpha\right]^T P \left[\int_{t-h}^{t} v(\alpha) d\alpha\right].$$

Lemma 2: (Projection Lemma [12]): Given a symmetric matrix Ψ and two matrices Λ and Γ of appropriate dimensions, the linear matrix inequality

$$\Psi + \Lambda^T \Theta^T \Gamma + \Gamma^T \Theta \Lambda < 0 \tag{10}$$

is feasible in matrix Θ if and only if

$$\mathcal{N}_{\Lambda}^{T} \Psi \mathcal{N}_{\Lambda} < 0 \tag{11}$$

and

$$\mathcal{N}_\Gamma^T \Psi \mathcal{N}_\Gamma < 0, \tag{12}$$

where $\mathcal{N}_\Lambda$ and $\mathcal{N}_\Gamma$ are any basis of the null space of Λ and Γ, respectively. For a matrix $\Gamma \in \mathbb{R}^{n\times m}$ with rank r, $\mathcal{N}_\Gamma \in \mathbb{R}^{(n-r)\times n}$ and satisfies two conditions $\mathcal{N}_\Gamma \Gamma = 0$ and $\mathcal{N}_\Gamma \mathcal{N}_\Gamma^T > 0$.

III. Stability and Performance Analysis of Time-delay LPV Systems

A. Stability Analysis

We first consider the unforced LPV system (8), that is

$$\dot{\bar{x}}(t) \;=\; \bar{A}(\rho(t))\bar{x}(t) + \bar{A}_h(\rho(t))\bar{x}(t - h(\rho(t))). \tag{13}$$

As the first result of this paper, we present the following theorem providing a sufficient condition to ensure asymptotic stability of the LPV system (13).

Theorem 1: The time-delay LPV system represented by (13) is asymptotically stable for all $0 \le h(\cdot) \le h_m$, if there exist continuously differentiable matrix function $P : \mathbb{R}^s \to \mathbb{S}_+^{2n\times 2n}$ and constant matrices $R, Q \in \mathbb{S}_+^{2n\times 2n}$ for all $\rho(t) \in \mathcal{F}_\mathcal{P}^v$ such that

$$\begin{bmatrix} \bar{A}^T P + P\bar{A} + \dot{P} + Q - R & P\bar{A}_h + R & h_m \bar{A}^T R \\ \star & -(1-\dot{h})Q - R & h_m \bar{A}_h^T R \\ \star & \star & -R \end{bmatrix} < 0. \tag{14}$$

Proof: Lyapunov-Krasovskii stability theory serves as a useful tool to derive delay-dependent conditions for the stability analysis of the systems We consider the following Lyapunov-Krasovskii functional

$$V(\bar{x}_t, \rho) = V_1(\bar{x}, \rho) + V_2(\bar{x}_t, \rho) + V_3(\bar{x}_t, \rho) \tag{15}$$

with

$$\begin{aligned} V_1(\bar{x}, \rho) &= \bar{x}^T(t) P(\rho(t)) \bar{x}(t) \\ V_2(\bar{x}_t, \rho) &= \int_{t-h(\rho(t))}^{t} \bar{x}^T(\xi) Q(\rho(\xi)) \bar{x}(\xi)\, d\xi \\ V_3(\bar{x}_t, \rho) &= \int_{-h_m}^{0} \int_{t+\theta}^{t} \dot{\bar{x}}^T(\xi)\; h_m R\; \dot{\bar{x}}(\xi)\, d\xi\, d\theta \end{aligned}$$

where the notation $\bar{x}_t(\theta)$ is used to represent $\bar{x}(t+\theta)$ for $\theta \in [-h_m \;\; 0]$. It is noted that (15) is chosen to be dependent on the LPV parameter vector $\rho(t)$ and the maximum delay h_m to result in less conservative stability conditions. In order for the system to be asymptotically stable, it suffices that time derivative of (15) along the trajectories of the system (13) is negative. One can readily obtain

$$\begin{aligned} \dot{V}_1(\bar{x}, \rho) &= \dot{\bar{x}}^T(t) P(\rho) \bar{x}(t) + \bar{x}^T(t) P(\rho) \dot{\bar{x}}(t) + \bar{x}^T(t) \dot{P}(\rho) \bar{x}(t), \\ \dot{V}_2(\bar{x}_t, \rho) &= \bar{x}^T(t) Q \bar{x}(t) + (1-\dot{h}) \bar{x}^T(t-h) Q \bar{x}(t-h), \end{aligned} \tag{16}$$

and

$$\dot{V}_3(\bar{x}_t, \rho) = h_m^2 \dot{\bar{x}}^T(t) R\; \bar{x}(t) - \int_{t-h_m}^{t} \dot{\bar{x}}^T(\theta)\; h_m R\; \dot{\bar{x}}(\theta) d\theta. \tag{17}$$

Since $h \le h_m$, the integral term in (17) satisfies

$$-\int_{t-h_m}^{t} \dot{\bar{x}}^T(\theta) h_m R \dot{\bar{x}}(\theta)\, d\theta \le -\int_{t-h}^{t} \dot{\bar{x}}^T(\theta) h_m R \dot{\bar{x}}(\theta) d\theta.$$

Employing Lemma 1, we have

$$-\int_{t-h}^{t} \dot{\bar{x}}^T(\theta) h_m R \dot{\bar{x}}(\theta)\, d\theta \le -\left[\bar{x}(t) - \bar{x}(t-h)\right]^T R \left[\bar{x}(t) - \bar{x}(t-h)\right].$$

Substituting for $\dot{\bar{x}}(t)$ and then collecting the terms yields

$$\begin{aligned} \dot{V}(\bar{x}_t, \rho) &\le \begin{bmatrix} \bar{x}(t) \\ \bar{x}(t-h) \end{bmatrix}^T \left(\mathcal{X} + \begin{bmatrix} \bar{A}^T \\ \bar{A}_h^T \end{bmatrix} h_m^2 R \begin{bmatrix} \bar{A}^T \\ \bar{A}_h^T \end{bmatrix}^T \right) \begin{bmatrix} \bar{x}(t) \\ \bar{x}(t-h) \end{bmatrix} \\ &= \begin{bmatrix} \bar{x}(t) \\ \bar{x}(t-h) \end{bmatrix}^T \left(\mathcal{X} + \begin{bmatrix} h_m \bar{A}^T R \\ h_m \bar{A}_h^T R \end{bmatrix} R^{-1} \begin{bmatrix} h_m \bar{A}^T R \\ h_m \bar{A}_h^T R \end{bmatrix}^T \right) \begin{bmatrix} \bar{x}(t) \\ \bar{x}(t-h) \end{bmatrix}, \end{aligned} \tag{18}$$

where

$$\mathcal{X} = \begin{bmatrix} \bar{A}^T P + P\bar{A} + \dot{P} + Q - R & P\bar{A}_h + R \\ \star & -(1-\dot{h})Q - R \end{bmatrix}.$$

To ensure that $\dot{V}(\bar{x}_t, \rho) < 0$ using (18), it is sufficient that (14) holds true.

B. Performance Analysis

Consider the state-space representation for the augmented LPV system (8) Next, we present the performance analysis condition for this time-delay LPV system.

Theorem 2: The LPV system (8) is asymptotically stable and satisfies $\|z\|_{\mathcal{L}_2} \le \gamma \|w\|_{\mathcal{L}_2}$ for $0 \le h(\cdot) \le h_m$ and zero initial condition if there exist a continuously differentiable matrix function $P : \mathbb{R}^s \to \mathbb{S}_+^{2n\times 2n}$, constant matrices $R, Q \in \mathbb{S}_+^{2n\times 2n}$ and a positive scalar γ for all $\rho(t) \in \mathcal{F}_\mathcal{P}^v$ such that

$$\begin{bmatrix} \bar{A}^T P + P\bar{A} + \dot{P} + Q - R & P\bar{A}_h + R & P\bar{B} & \bar{C}^T & h_m \bar{A}^T R & 0 \\ \star & -(1-\dot{h})Q - R & 0 & \bar{C}_h^T & h_m \bar{A}_h^T R & 0 \\ \star & \star & -\gamma I & \bar{D}^T & h_m \bar{B}^T R & 0 \\ \star & \star & \star & \star & -\gamma I & 0 \\ \star & \star & \star & \star & \star & -R \end{bmatrix} < 0. \tag{19}$$

Proof: We first define a Lyapunov-Krasovskii functional similar to the one introduced in Theorem 1. Next, we apply the following congruent transformation

$$\mathcal{T} = \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \end{bmatrix}$$

to (19). In the obtained inequality, it can be observed that the negative definiteness of the upper left 3×3 block matrix, in light of Theorem 1, concludes the asymptotic stability of the system (8). Applying Schur complement to (19) twice results in

$$\begin{bmatrix} \bar{A}^T P + P\bar{A} + \dot{P} + Q - R & P\bar{A}_h + R & P\bar{B} \\ \star & -(1-\dot{h})Q - R & 0 \\ \star & \star & -\gamma I \end{bmatrix} + \begin{bmatrix} \bar{C}^T \\ \bar{C}_h^T \\ \bar{D}^T \end{bmatrix} \gamma^{-1} \begin{bmatrix} \bar{C}^T \\ \bar{C}_h^T \\ \bar{D}^T \end{bmatrix}^T + \begin{bmatrix} \bar{A}^T \\ \bar{A}_h^T \\ \bar{B}^T \end{bmatrix} h_m^2 R \begin{bmatrix} \bar{A}^T \\ \bar{A}_h^T \\ \bar{B}^T \end{bmatrix}^T < 0. \tag{20}$$

Multiplying the above inequality from left and right by $[\bar{x}^T(t) \quad \bar{x}^T(t-\tau) \quad w^T(t)]^T$ and its transpose, respectively, followed by minor algebraic manipulations yields

$$\begin{aligned} &\dot{\bar{x}}^T(t)P\bar{x}(t) + \bar{x}^T(t)P\dot{\bar{x}}(t) + \bar{x}^T(t)\dot{P}\bar{x}(t) \\ &+ \bar{x}^T(t)Q\bar{x}(t) + (1-\dot{h})\bar{x}^T(t-h)\ Q\ x(t-h) \\ &+ h_m^2\dot{\bar{x}}^T(t)R\dot{\bar{x}}(t) - [\bar{x}(t) - \bar{x}(t-h)]^T R\,[\bar{x}(t) - \bar{x}(t-h)] \\ &- \gamma w^T(t)w(t) + \frac{1}{\gamma}z^T(t)z(t) < 0. \end{aligned}$$

Referring to (15), we obtain

$$\dot{V}(\bar{x}_t, \rho) - \gamma w^T(t)w(t) + \frac{1}{\gamma}z^T(t)z(t) < 0. \tag{21}$$

Integrating both sides of the inequality (21) from 0 to ∞ and using $V|_{t=0} = V|_{t=\infty} = 0$ (due to the augmented system asymptotic stability and zero initial condition for the system states), we arrive at $\|z\|_{\mathcal{L}_2} \le \gamma\|w\|_{\mathcal{L}_2}$ and this completes the proof. ■

C. Employing Slack Variables

In order to establish a synthesis condition for the filter design, the corresponding system matrices (8) are substituted in (19); this, however, results in a bilinear matrix inequality problem due to the byproduct of the filter matrices with the unknown matrix function P and unknown matrix R. Therefore, we will seek an alternative method based on the introduction of *slack variables* to reformulate the corresponding problem to ensure that an LMI condition is achieved. The following lemma provides the solution.

Lemma 3: The LPV system (8) is asymptotically stable for all $h(\cdot) \le h_m$ and satisfies $\|z\|_{\mathcal{L}_2} < \gamma\|w\|_{\mathcal{L}_2}$ if there exist a continuously differentiable matrix function $P : \mathbb{R}^s \to \mathbb{S}_+^{2n\times 2n}$, constant matrices R, Q, V_1, V_2, $V_3 \in \mathbb{S}_+^{2n\times 2n}$ and a positive scalar γ for any admissible parameter trajectory $\rho(t) \in \mathcal{F}_{\mathcal{P}}^v$, such that

$$\begin{bmatrix} -V_1 - V_1^T & P - V_2^T + V_1\bar{A} & -V_3^T + V_1\bar{A}_h & V_1\bar{B} & 0 & h_mR \\ \star & \dot{P} + Q - R + \bar{A}^TV_2^T + V_2\bar{A} & R + \bar{A}^TV_3^T + V_2\bar{A}_h & V_2\bar{B} & \bar{C}^T & 0 \\ \star & \star & -(1-\dot{h})Q - R + \bar{A}_h^TV_3^T + V_3\bar{A}_h & V_3\bar{B} & \bar{C}_h^T & 0 \\ \star & \star & \star & -\gamma I & \bar{D}^T & 0 \\ \star & \star & \star & \star & -\gamma I & 0 \\ \star & \star & \star & \star & \star & -R \end{bmatrix} < 0. \tag{22}$$

Proof: We start with rewriting (22) as $\Psi + \Lambda^T\Theta^T\Gamma + \Gamma^T\Theta\Lambda < 0$, with

$$\Psi = \begin{bmatrix} 0 & P & 0 & 0 & 0 & h_mR \\ \star & \dot{P} + Q - R & R & 0 & \bar{C}^T & 0 \\ \star & \star & -(1-\dot{h})Q - R & 0 & \bar{C}_h & 0 \\ \star & \star & \star & -\gamma I & \bar{D}^T & 0 \\ \star & \star & \star & \star & -\gamma I & 0 \\ \star & \star & \star & \star & \star & -R \end{bmatrix},$$

$$\Lambda = \begin{bmatrix} -I & \bar{A} & \bar{A}_h & \bar{B} & 0 & 0 \end{bmatrix}, \quad \Theta = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix},$$

$$\Gamma = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \end{bmatrix}. \tag{23}$$

The matrix variables V_1, V_2 and V_3 are known as slack variables [13]. We next use Lemma 2 by finding the bases for the null space of Λ and Γ as

$$\mathcal{N}_\Lambda = \begin{bmatrix} \bar{A} & \bar{A}_h & \bar{B} & 0 & 0 \\ I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix}, \qquad \mathcal{N}_\Gamma = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}.$$

We then substitute the two matrices above in the solvability conditions of Lemma 2. Using the solvability condition (11) results in the LMI condition (22). On the other hand, the second solvability condition, i.e., (12), leads to the following LMI

$$\begin{bmatrix} -\gamma I & \bar{D}^T & 0 \\ \star & -\gamma I & 0 \\ \star & \star & -R \end{bmatrix} < 0, \tag{24}$$

which is part of LMI (22) and is always satisfied as long as there is a feasible solution to (22). In summary, feasibility of the LMI condition (22) ensures that the LMI problem (19) is feasible and based on Theorem 2, the proof of Lemma 3 is complete. ■

Remark 1: The choice of slack variables in (23) is inspired by the authors' previous work [17].

IV. Filter Design for LPV Systems

In order to find the filter matrices, we employ Lemma 3 for the augmented system (8). The following theorem summarizes the procedure for the filter design.

Theorem 3: If there exist a parameter-dependent continuously differentiable matrix $\tilde{P}(\rho) : \mathbb{R}^s \to \mathbb{S}_+^{2n\times 2n}$, parameter-dependent matrices $X(\rho)$, $Y(\rho) : \mathbb{R}^s \to \mathbb{S}_+^{n\times n}$, constant matrices $\tilde{R}, \tilde{Q} \in \mathbb{S}_+^{2n\times 2n}$, parameter-dependent matrices $\hat{A}(\rho), \hat{A}_h(\rho), \hat{B}(\rho), \hat{C}(\rho), \hat{C}_h(\rho)$ and $D_F(\rho)$, two given scalars λ_2, $\lambda_3 \in \mathbb{R}$ and a positive scalar γ for any admissible parameter trajectory $\rho(t) \in \mathcal{F}_{\mathcal{P}}^v$, such that

$$\begin{bmatrix} -2\tilde{V} & \tilde{P} - \lambda_2\tilde{V} + \tilde{A} & -\lambda_3\tilde{V} + \tilde{A}_h \\ \star & \dot{\tilde{P}} + \tilde{Q} - \tilde{R} + \lambda_2(\tilde{A} + \tilde{A}^T) & \tilde{R} + \lambda_3\tilde{A}^T + \lambda_2\tilde{A}_h \\ \star & \star & -(1-\dot{h})\tilde{Q} - \tilde{R} + \lambda_3(\tilde{A}_h + \tilde{A}_h^T) \\ \star & \star & \star \\ \star & \star & \star \\ \star & \star & \star \end{bmatrix}$$

$$\left.\begin{matrix} \tilde{B} & 0 & h_m\tilde{R} \\ \lambda_2\tilde{B} & \tilde{C}^T & 0 \\ \lambda_3\tilde{B} & \tilde{C}_h^T & 0 \\ -\gamma I & \tilde{D}^T & 0 \\ \star & -\gamma I & 0 \\ \star & \star & -\tilde{R} \end{matrix}\right] < 0, \tag{25}$$

where the relationship between the matrices with ˆ and those with ˜ is given by (31), then there exist a filter in the form of (3) such that the estimation error system is asymptotically stable and satisfies $\|z\|_{\mathcal{L}_2} < \gamma\|w\|_{\mathcal{L}_2}$ for $0 \leq h(\cdot) \leq h_m$ and all $\rho(t) \in \mathcal{F}_{\mathcal{P}}^{v}$. In addition, such a filter matrices are obtained as follows.

1- Solve M and N from the factorization problem

$$I - XY = NM^T.$$

2- Find the filter matrices as

$$\begin{aligned} B_F &= N^{-1}\hat{B} \\ A_F &= N^{-1}(\hat{A} - XAY - NB_FC_2Y)M^{-T} \\ A_{hF} &= N^{-1}(\hat{A}_h - XA_hY - NB_FC_{2h}Y)M^{-T} \\ C_F &= (\hat{C} - D_FC_2Y)M^{-T} \\ C_{hF} &= (\hat{C}_h - D_FC_{2h}Y)M^{-T}. \end{aligned} \tag{26}$$

Proof: In the matrix inequality (22), we substitute the augmented system matrices $\bar{A}$, $\bar{A}_h$, $\bar{B}_1$, $\bar{C}_1$ and $\bar{C}_{1h}$ from (8) and select the three slack to be as $V_1 = V$, $V_2 = \lambda_2 V$ and $V_3 = \lambda_3 V$ for scalars λ_2 and λ_3. Next, we partition the slack matrix V as

$$V = \begin{bmatrix} X & N \\ N^T & G \end{bmatrix}. \tag{27}$$

Define

$$V^{-1} = \begin{bmatrix} Y & M \\ M^T & H \end{bmatrix}, \tag{28}$$

which yields $XY + NM^T = I$. Next, we perform the congruent transformation $\mathcal{T} = diag(Z^T, Z^T, Z^T, I, I, Z^T)$ on (22) with

$$Z = \begin{bmatrix} Y & I \\ M^T & 0 \end{bmatrix}. \tag{29}$$

Consequently, we obtain the following matrix inequality

$$\left[\begin{matrix} -2\tilde{V} & \tilde{P} - \lambda_2\tilde{V} + \tilde{A} \\ \star & \dot{\tilde{P}} + \tilde{Q} - \tilde{R} + \lambda_2(\tilde{A} + \tilde{A}^T) \\ \star & \star \\ \star & \star \\ \star & \star \\ \star & \star \end{matrix}\right.$$

$$\left.\begin{matrix} -\lambda_3\tilde{V} + \tilde{A}_h & \tilde{B} & 0 & h_m\tilde{R} \\ \tilde{R} + \lambda_3\tilde{A}^T + \lambda_2\tilde{A}_h & \lambda_2\tilde{B} & \tilde{C}^T & 0 \\ -(1-\dot{h})\tilde{Q} - \tilde{R} + \lambda_3(\tilde{A}_h + \tilde{A}_h^T) & \lambda_3\tilde{B} & \tilde{C}_h^T & 0 \\ \star & -\gamma I & \bar{D}^T & 0 \\ \star & \star & -\gamma I & 0 \\ \star & \star & \star & -\tilde{R} \end{matrix}\right] < 0, \tag{30}$$

where

$$\tilde{V} = Z^TVZ = \begin{bmatrix} Y & I \\ I & X \end{bmatrix},$$

and

$$\tilde{P} = Z^TPZ, \quad \dot{\tilde{P}} = Z^T\dot{P}Z, \quad \tilde{R} = Z^TRZ, \quad \tilde{Q} = Z^TQZ.$$

In addition, the plant-related matrices are obtained as

$$\begin{aligned} \tilde{A} &= Z^TV\bar{A}Z = \begin{bmatrix} AY & A \\ \hat{A} & XA + \hat{B}C_2 \end{bmatrix}, \\ \tilde{A}_h &= Z^TV\bar{A}_hZ = \begin{bmatrix} A_hY & A_h \\ \hat{A}_h & XA_h + \hat{B}C_{2h} \end{bmatrix}, \\ \tilde{B} &= Z^TV\bar{B} = \begin{bmatrix} B_1 \\ XB_1 + \hat{B}D_{21} \end{bmatrix}, \\ \tilde{C} &= \bar{C}Z = \begin{bmatrix} C_1Y - \hat{C} & C_1 - D_FC_2 \end{bmatrix}, \\ \tilde{C}_h &= \bar{C}_hZ = \begin{bmatrix} C_{1h}Y - \hat{C}_h & C_{1h} - D_FC_{2h} \end{bmatrix}. \end{aligned} \tag{31}$$

In the above equations, we have used the variable changes as

$$\begin{aligned} \hat{A} &= XAY + NB_FC_2Y + NA_FM^T \\ \hat{A}_h &= XA_hY + NB_FC_{2h}Y + NA_{Fh}M^T \\ \hat{B} &= NB_F \\ \hat{C} &= D_FC_2Y + C_FM^T \\ \hat{C}_h &= D_FC_{2h}Y + C_{hF}M^T. \end{aligned} \tag{32}$$

Finally, by reversing the transformations in (32), the filter matrices are obtained as in (26). ■

Remark 2: In LMI (30), the (2,2)-entry includes a derivative term that can be replaced by $\dot{\tilde{P}} = \frac{\partial\tilde{P}}{\partial\rho}\dot{\rho}$. Due to the affine dependency of this matrix inequality on $\dot{\rho}$, it is only required to solve this feasibility problem at vertices of $\dot{\rho}$. Therefore, in this matrix inequality, one can replace the term $\dot{\tilde{P}}$ with $\sum_{i=1}^{s} \pm \left(v_i \frac{\partial\tilde{P}}{\partial\rho}\right)$ [14].

Remark 3: The LMI problem (30) is infinite-dimensional due to the dependency of the system matrices on LPV parameters continuously. A standard approach to solve the parameterized LMIs like (30) is to initially select some basis functions to represent the dependency of the matrix variables on the LPV parameters and then grid the parameter space. Finally, the obtained finite-dimensional LMI problem is solved at the grid points and then checked on a finer grid [2]. We may impose various structures on the LMI variables. In this paper we consider

$$X(\rho) = X_0 + \rho(t)X_1 + \frac{1}{2}\rho^2(t)X_2 + \ldots \tag{33}$$

and similarly for the other variables.

V. Simulation Results

In this section, we present an illustrative example to show the effectiveness of the proposed filtering method. We also compare the results of this work with [6] that offers the least conservative estimation results for the case of time-varying delays in the literature. Consider a time-delayed LPV system described by

$$
\begin{aligned}
\dot{x}(t) &= \begin{bmatrix} 0 & 1+0.2\sin(t) \\ -2 & -3+0.1\sin(t) \end{bmatrix} x(t) \\
&+ \begin{bmatrix} 0.2\sin(t) & 0.1 \\ -0.2+0.1\sin(t) & -0.3 \end{bmatrix} x(t-h(t)) + \begin{bmatrix} -0.2 \\ -0.2 \end{bmatrix} w(t), \\
z(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0.5 & 0 \end{bmatrix} x(t-h(t)), \\
y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) + 0.3w(t).
\end{aligned} \tag{34}
$$

We aim to estimate the output signal $z(t)$ using the measurements from the noisy signal $y(t)$. We assume that the *sine* term in the above model corresponds to the LPV parameter, i.e., $\rho(t) = \sin(t)$, whose functional representation is not known *a priori* but it can be measured in real time. It is apparent that the parameter space and its rate belong to $[-1\ 1]$. Also the associated time-varying delay is considered to be parameter dependent satisfying $h(\rho(t)) = \lambda|\sin(t)|$, where λ is the magnitude of the periodic time delay. Next we employ Theorem 3 to design the parameter-varying filter matrices. Table I shows the worst case energy-to-energy gain of the estimation error system, versus magnitude of the periodic time delay λ, using the proposed approach in this paper and the one in [6]. It is apparent that the performance of the estimation has been improved. Also, it is observed that for $\lambda \geq 1$, where $\dot{h} \geq 1$, the method in [6] fails, whereas the current study can handle this case effectively. Two scalars λ_2 and λ_3 in Theorem 3 were chosen to be $\lambda_2 = 2$ and $\lambda_3 = 7$ through trial and errors; however, these additional design parameters can be optimized using a 2-D search to achieve an even further improved performance.

TABLE I

INDUCED $\mathcal{L}_2$-GAIN (γ) FOR VARIOUS DELAYS USING THE RESULTS OF THIS PAPER AND THAT IN [6]

λ	0.1	0.5	0.9	1	2	2.5
Ref. [6]	0.29	0.3	0.45	infeasible	infeasible	infeasible
Theorem 3	0.04	0.12	0.34	0.54	1.3	6.7

Shown in Figure 1 is the time simulation of the signal $z(t)$ along with the estimated signal corresponding to the filter obtained from Theorem 3 (signal z_1) and the method in [6] (signal z_2). The magnitude of time delay is considered to be $\lambda = 0.5$ and the disturbance signal is a pulse shown in Figure 1. It is observed that Theorem 3 provides a better estimation of signal $z(t)$ even in the presence of an external disturbance signal.

VI. Concluding Remarks

In this study, a new filtering design method for LPV systems with state delay is presented. With an appropriate choice of Lyapunov-Krasovskii functional, a synthesis condition is obtained that is delay- and rate-dependent. By utilizing slack variables, the synthesis conditions are expressed in terms of linear matrix inequality (LMI) conditions, which can be solved efficiently using the existing computational tools. Compared with the existing methods in the literature,

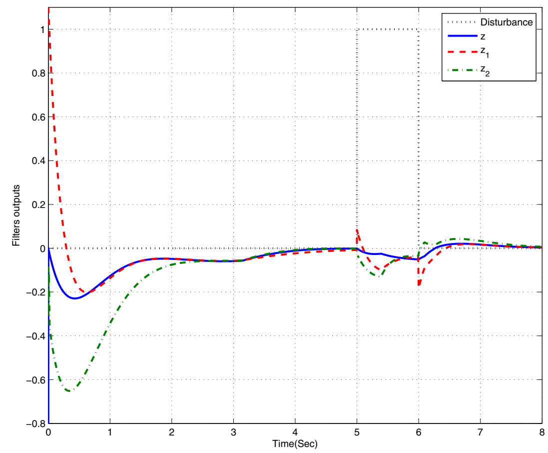

Fig. 1. Comparison between z_1 (this study) and z_2 (result of [6])

the proposed method is shown to provide less conservative results.

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