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Nonlinear Model Order Reduction of Burgers' Equation Using Proper Orthogonal Decomposition

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Abstract— **In this paper, we examine a model order reduction approach for dynamic systems governed by Burgers' equation with Neumann boundary conditions. The proper orthogonal decomposition (POD) method is employed here that provides a reliable and accurate modeling approach, while the temporal discretization of the continuous error function leads to a more accurate estimation of the defined cost function. We will investigate the accuracy of the reduced-order model compared to the finite element (FE) model by choosing an adequate number of basis functions for the approximating subspace. The derived lumped-parameter model for Burgers' equation is then described by a nonlinear state-space model. We finally demonstrate the accuracy of the reduced-order model through a numerical example, where we show that a 7-dimensional POD can accurately estimate the system output.**

I. Introduction

Computational modeling and simulation of nonlinear complex, turbulent systems implementing the standard discretization schemes like finite element or finite difference, may require a large number of degrees of freedom to accurately describe the fluid flows. Consequently, the spatial discretization leads to scarce, but substantial nonlinear systems of ordinary differential equations (ODEs) that approximate the solution of the given system. However, with respect to both storage and computing time, these methods are inefficient. This can be crucial when the real-time solutions of complex systems in feedback control synthesis are required. As a remedy, the reduced-order modeling was introduced to describe the original mathematical model by a smaller model in a way that it can still represent certain significant aspects of the system or process with a good accuracy, depending on the order of the reduced model. That is to say, in implementing different model order reduction schemes, the lowest order of the reduced model, which accurately approximates the original system is desired. To achieve this, the original system or process should be described by a number of basis functions that are extracted from the expected solution of the system.

The proper orthogonal decomposition (POD), also known as the Karhunen-Loeve decomposition, can provide us with an effective tool based on projecting the dynamical system onto subspaces of basis elements that express characteristics of the given system. This is in contrast to, e.g., finite element techniques, where the elements are not correlated to the physical properties of the system they approximate [1], [2].

The implementation of the model order reduction approaches was originally developed by [3], [4], [5] in the framework of the structural simulation and later in simulation of incompressible viscous flows [6]. Among several commonly used model reduction techniques like balanced truncation and singular value decomposition based methods, the POD has received much attention in recent years as a tool to analyze complex physical systems [7], [8]. It was adopted by [9] to study turbulent flows. Another application of POD has been in the field of time-dependent partial differential equations (PDEs), where the snapshots are taken on a certain grid of time instants. It has been also successfully applied in different fields including signal analysis and pattern recognition [16], fluid dynamics and coherent structures [9], and more recently in optimal control of evolution problems [7].

The basis functions extracted by POD can be used in a collocation formulation of Galerkin projection that leads to a finite dimensional system with the smallest possible degrees of freedom. Therefore, the POD Galerkin technique is well suited in optimal control synthesis and the estimation of parameters in systems described by PDEs [11], [12]. The POD Galerkin scheme has been also extended separately for elliptic PDEs in [13]. Moreover, the application of POD Galerkin schemes for the spatial approximation has been substantially studied in [14], [15].

In the present work, we focus on the continuous POD (as opposed to snapshot POD) method and its application for the model order reduction of the forced Burgers' equation, which has characteristics similar to the Navier-Stokes equations. The objective of this work is to take advantage of the underlying characteristics of the continuous POD method to reduce the original model to a number of ODEs that would then be represented in the state-space form.

Throughout the paper, unless otherwise specified, we use the notation $\langle .,. \rangle$ to show the inner product of the given basis functions in the finite element method, representing the spatial domain integration of the product of the given basis functions. Also, $\mathcal{W}_i^l$ represents the i^{th} Fourier coefficient of the reduced model of the order l and $\mathbb{R}^m$ is an m-dimensional Euclidean space. In addition, we define $A \circ B$ as the Hadamard product of the matrices A and B, of the same dimension, $m \times n$ such that $[A \circ B]_{ij} = [A_{ij}][B_{ij}]$.

The rest of the paper is organized as follows: Section II describes the characteristics of the Burgers' PDE. Section III explains FEM and primary discretization of the system. The continuous POD and the fundamental idea behind it will be described in Section IV. Section V provides the obtained reduced-order models and their state-space representation. Finally, the simulation results for the given numerical example are discussed in section VI.

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II. Nonlinear Parabolic Model of Burgers' Equation

Over the past three decades, Burgers' equation has been used for the better understanding of turbulence and other nonlinear phenomena as the very important parts of complex systems. This nonlinear parabolic partial deferential equation (PDE) provides a precise model for investigating different control problems such as boundary and distributed parameter feedback control problems. In the present study, we consider this nonlinear PDE model with Neumann boundary conditions aiming at developing a reduced-order and control-oriented model. In fact, we reduce this nonlinear PDE model to a number of ordinary differential equations (ODE), and then represent the system in state-space form using proper orthogonal decomposition (POD) method and finite element models (FEMs).

Suppose that Ω represents the spatial interval $(0, L)$ and that for $T > 0$, we set $Q = (0, T) \times \Omega$. For a given velocity $w(t, x)$ and viscosity ν, the governing viscous Burgers' PDE and the initial and boundary conditions are

$$\frac{\partial w(t,x)}{\partial t} + w(t,x)\frac{\partial w(t,x)}{\partial x} - \nu \frac{\partial^2 w(t,x)}{\partial x^2} = f(t,x), \quad (1)$$

$$I.C: \quad w(0,x) = w_0(x), \quad (2)$$

$$B.C: \quad w_x(0,t) = u_1(t), w_x(L,t) = u_2(t) \quad (3)$$

where $(t, x) \in Q$, $u_1(t)$ and $u_2(t)$ are the changing boundary conditions or the system inputs, and the viscosity ν is considered as $1/Re$, where Re represents the Reynolds number. The function f is the forcing term that is assumed to be square integrable in space and time. We define the Hilbert space of Lebesgue square integrable functions as $H = L^2(\Omega)$. We note that the function f is in H if it satisfies

$$\int_0^T \|f(t,x)\|_H^2 \mathrm{d}t < \infty. \quad (4)$$

III. Order Reduction of Burgers' Equation with Finite Element Method

The finite element method (FEM) is considered as a general method to approximate partial deferential equations with lumped-parameter (ODE) models. An advantage of this technique over other methods is that if PDE is time dependent, then it can be reduced to a system of ODEs which can be integrated using existing techniques. Having a system of linear or nonlinear ODEs can allow to represent the system in the linear or nonlinear state-space form, which would be helpful for the control synthesis purposes.

FEM representation of the Burgers' equation

In order to discretize the Burgers' equation in the spatial domain, the interval is divided into N subintervals $[x_j, x_{j+1}]$ and we define $h_j = x_{j+1} - x_j$. We assume all the elements are of equal size (uniformly spaced mesh) and hence $h_1 = \ldots = h_N = h$. Therefore, the FEM basis functions are defined as [17]

(i) for elements $e_j,\ j = 1 : N-1$,

$$\mathcal{N}_j(x) = \begin{cases} \frac{x - x_{j-1}}{h}, & x_{j-1} \le x \le x_j \\ \frac{x_{j+1} - x}{h}, & x_j \le x \le x_{j+1} \\ 0 & \text{otherwise} \end{cases} \quad (5)$$

(ii) for element e_0,

$$\mathcal{N}_0(x) = \begin{cases} \frac{x_1 - x}{h}, & 0 \le x \le x_1 \\ 0 & \text{otherwise} \end{cases} \quad (6)$$

(iii) for element e_N

$$\mathcal{N}_N(x) = \begin{cases} \frac{x - x_{N-1}}{h}, & x_{N-1} \le x \le x_N \\ 0 & \text{otherwise.} \end{cases} \quad (7)$$

The approximation of $w(t, x)$ in the space spanned by the piecewise linear basis functions is given by

$$w^N(t,x) = \sum_{i=0}^{N} \mathcal{W}_i(t)\mathcal{N}_i(x), \quad (8)$$

where $\mathcal{W}_i(t)$ is the nodal value, i.e., $w(t, x_i)$ at the ith node and time t [17]. The *weak solution* approach is employed by multiplying both sides of (1) by a piecewise smooth test function $v(x)$ and integrating in the spatial variable's domain [18]. Taking the integral from both sides and also substituting the second order derivative term by the chain rule results in

$$\int_0^L \left(w_t(t,x) + \frac{1}{2}[w^2(t,x)]_x \right) v(x)\,\mathrm{d}x - \nu \left[u_2(t)v(L) - u_1(t)v(0) - \int_0^L w_x(t,x)v'(x)\,\mathrm{d}x \right] = \int_0^L f(t,x)v(x)\,\mathrm{d}x. \quad (9)$$

Using the group finite element (GFE) method proposed in [10], nonlinear term can be approximated as

$$w^2(t,x) \approx \sum_{i=0}^{N} \mathcal{W}_i^2(t)\mathcal{N}_i(x). \quad (10)$$

Since $v(x)$ is arbitrary and piecewise smooth, we let $v(x) = \mathcal{N}_j(x)$, for $j = 0, 1, \ldots, N$. Using Galerkin method and substituting (10) into (9) yields

$$\sum_{i=0}^{N} \dot{\mathcal{W}}_i(t) \int_0^L \mathcal{N}_i(x)\mathcal{N}_j(x)\,\mathrm{d}x + \frac{1}{2}\sum_{i=0}^{N} \mathcal{W}_i^2(t) \int_0^L \mathcal{N}_i'(x)\mathcal{N}_j(x)\,\mathrm{d}x - \nu \left[u_2(t)\mathcal{N}_j(L) - u_1(t)\mathcal{N}_j(0) \right] - \nu \sum_{i=0}^{N} \mathcal{W}_i(t) \int_0^L \mathcal{N}_i'(x)\mathcal{N}_j'(x)\,\mathrm{d}x = \int_0^L f(t,x)\mathcal{N}_j(x)\,\mathrm{d}x.$$

We consider the following notation

$$M_{ij} = \langle \mathcal{N}_i(x), \mathcal{N}_j(x) \rangle, S_{ij} = \langle \mathcal{N}_i'(x), \mathcal{N}_j'(x) \rangle, \quad F_j(t) = \langle f(t,x), \mathcal{N}_j(x) \rangle. \quad (11)$$

For the nonlinear term, we represent it as

$$(N(\mathcal{W}(t)))_j = \frac{1}{2}\sum_{i=0}^{N}\mathcal{W}_i^2(t)\int_0^L \mathcal{N}_i'(x)\mathcal{N}_j(x),$$
$$K_{ij} = \langle \mathcal{N}_i'(x), \mathcal{N}_j(x)\rangle,$$

Therefore

$$N(\mathcal{W}(t)) = \frac{1}{2}K\mathcal{W}^2(t), \tag{12}$$

where $\mathcal{W}^2(t) = [\mathcal{W}_0^2(t) \ \dots \ \mathcal{W}_N^2(t)]^\top$. Finally, the reduced-order model with input vector $U(t) = [u_1(t) \quad u_2(t)]^\top$ is described as a set of $N+1$ ordinary differential equations

$$M\dot{\mathcal{W}}(t) + \nu S\mathcal{W}(t) + \frac{1}{2}K\mathcal{W}^2(t) - \nu LU(t) = F(t), \tag{13}$$

where

$$L = \begin{bmatrix} -1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{bmatrix}^\top_{(N+1)\times 2}.$$

To solve this set of nonlinear ODEs, we need to specify the initial condition. To do so, the given initial condition should be described in the space spanned by the basis functions, i.e.,

$$w_0(x) \approx w^N(0,x) = \sum_{i=0}^{N}\mathcal{W}_i(0)\mathcal{N}_i(x). \tag{14}$$

By multiplying both sides of the equality by the test function and again using the weak solution approach, we obtain

$$\sum_{i=0}^{N}\mathcal{W}_i(0)\int_0^L \mathcal{N}_i(x)\mathcal{N}_j(x) = \int_0^L w_0(x)\mathcal{N}_j(x), \quad j = 0,\dots,N. \tag{15}$$

This can be represented in the matrix form as

$$M\mathcal{W}(0) = \mathcal{P}, \tag{16}$$

where $\mathcal{P}_j = \langle w_0(x), \mathcal{N}_j(x)\rangle$. This linear equation gives the initial conditions for the $N+1$ ODEs described by (13).

IV. Proper Orthogonal Decomposition Method

The fundamental idea behind proper orthogonal decomposition (POD) is to optimally represent the system in a mean-squared error sense using an orthonormal basis of rank l. Let $Y = [y_1,\dots,y_n]_{m\times n}$ be a real data matrix containing the n snapshot vectors of m spatial data points. The POD basis of rank l is optimal in the sense of representing the columns $\{y_j\}_{j=1}^n$ of Y as a linear combination by an orthonormal basis of rank l [1]. We endow the Euclidean space $\mathbb{R}^m$ with the weighted inner product as

$$\begin{aligned}\langle u,\tilde{u}\rangle_W &= u^T W\tilde{u} = \langle u, W\tilde{u}\rangle_{\mathbb{R}^m} \\ &= \langle Wu,\tilde{u}\rangle_{\mathbb{R}^m} \quad \text{for} \ \ u,\tilde{u}\in\mathbb{R}^m\end{aligned} \tag{17}$$

where $W \in \mathbb{R}^{m\times m}$ is a symmetric, positive-definite matrix. Note that the vector $y(t)$, $t\in[0,T]$, now represents a function in Ω evaluated at m grid points. Therefore, we should supply $\mathbb{R}^m$ by a weighted inner product representing a discretized inner product in an appropriate function space. Since, the mass matrix in (13) is symmetric, real and positive definite, it can be considered as the weight matrix in the predefined inner product. The goal is to determine a POD basis of rank $l \le n$ that gives the best estimate of the entire trajectory $\nu^y = \text{span}\{y(t)|t\in[0,T]\} \subset \mathbb{R}^m$. The optimality is achieved by minimizing the error between the data and its projection onto the basis set [14]

$$\begin{aligned}J = \min \int_0^T \|y(t) - \sum_{i=1}^{l}\langle y(t),\tilde{u}_i\rangle_M \tilde{u}_i\|_M^2 \,\mathrm{d}t \\ \text{s.t.} \quad \langle \tilde{u}_i,\tilde{u}_j\rangle_M = \delta_{ij} \quad \text{for } 1\le i,j\le l.\end{aligned} \tag{18}$$

Since the entire trajectory is not available in practical computation, we suppose that we know the solution of (13) at the given time instants t_j, $j = 1,\dots,n$. This translates to minimizing the following cost function J while the constraints are met [13]

$$\begin{aligned}J = \min \sum_{j=1}^{n}\alpha_j\|y_j - \sum_{i=1}^{l}\langle y_j,\tilde{u}_i\rangle_M\tilde{u}_i\|_M^2 \\ \text{s.t.} \quad \langle \tilde{u}_i,\tilde{u}_j\rangle_M = \delta_{ij} \quad \text{for } 1\le i,j\le l,\end{aligned} \tag{19}$$

where α_j's denote the non-negative trapezoidal weights defined by

$$\alpha_1 = \frac{\Delta t}{2}, \ \alpha_j = \Delta t \quad \text{for } 2\le j\le n-1, \ \alpha_n = \frac{\Delta t}{2}. \tag{20}$$

To solve the above constrained optimization problem, first-order necessary optimality condition is applied. Therefore, the associated Lagrange functional is described as

$$\mathcal{L} : \underbrace{\mathbb{R}^m \times \dots \times \mathbb{R}^m}_{l-times} \times \mathbb{R}^{l\times l}$$

$$\begin{aligned}\mathcal{L}(u_1,\dots,u_l,\Lambda) = \sum_{j=1}^{n}\alpha_j\|y_j - \sum_{i=1}^{l}\langle y_j,u_i\rangle_M u_i\|_M^2 \\ + \sum_{i=1}^{l}\sum_{j=1}^{l}\Lambda_{ij}(1 - \langle u_i,u_j\rangle_M),\end{aligned} \tag{21}$$

where $u_1,\dots,u_l \in \mathbb{R}^m$ and $\Lambda\in\mathbb{R}^{l\times l}$. First-order necessary optimality condition gives

$$\Delta_{u_i}\mathcal{L}(u_1,\dots,u_l,\Lambda) = 0 \ \text{ in } \mathbb{R}^m, \ 1\le i\le l. \tag{22}$$

Also noting that

$$\langle u_i,u_j\rangle_M = \delta_{ij} \quad \text{for } 1\le i,j\le l, \tag{23}$$

the following is derived from the optimality condition (22)

$$YDY^TMu_i = \lambda_i u_i \ \text{ for } i = 1,\dots,l, \tag{24}$$

where $D = \text{diag}(\alpha_1,\dots,\alpha_n) \in \mathbb{R}^{n\times n}$. By defining $u_i = M^{-1/2}\bar{u}_i$ in (24) and multiplying it by $M^{1/2}$ from left, we obtain

$$M^{1/2}YDY^TM^{1/2}\bar{u}_i = \lambda_i\bar{u}_i \ \text{ for } i = 1,\dots,l. \tag{25}$$

Considering (23), we have

$$\begin{aligned}\langle \bar{u}_i, \bar{u}_j\rangle_{\mathbb{R}^m} &= \bar{u}_i^T \bar{u}_j = u_i^T M u_j \\ &= \langle u_i, u_j\rangle_M = \delta_{ij} \quad \text{for } 1 \le i, j \le l.\end{aligned} \tag{26}$$

Defining $\bar{Y} = M^{1/2} Y D^{1/2} \in \mathbb{R}^{m\times n}$ and knowing that $M^T = M$ and $D^T = D$, the solution to the optimization problem (19) is obtained by solving the symmetric $m \times m$ eigenvalue problem

$$\begin{aligned}\bar{Y}\bar{Y}^T \bar{u}_i = \lambda_i \bar{u}_i, \ \ 1 \le i \le l, \\ \langle \bar{u}_i, \bar{u}_j\rangle_{\mathbb{R}^m} = \delta_{ij}, \ \ 1 \le i, j \le l.\end{aligned} \tag{27}$$

The choice of the number of basis functions l, leading to an accurate description of the original model, is certainly of critical importance for applying POD. There is no general rule for the selection of l; it is rather based on heuristic considerations along with observing the captured relative energy by the basis functions [19], which is expressed by

$$\mathcal{E}(l) = \frac{\sum_{i=1}^{l} \lambda_i}{\sum_{i=1}^{d} \lambda_i}. \tag{28}$$

where $d = rank(\bar{Y})$.

V. Development of the Reduced-order Model Using POD

In this section, the derivation of the reduced order model of the Burgers' equation using POD method is described. To obtain a control-oriented model, we use the approximation of $w(t,x)$ in the space spanned by the POD basis functions $\psi_i(x), \ i = 1, \dots, l$ as

$$w(t,x) = \sum_{i=1}^{l} \langle w(t,x), \psi_i(x)\rangle_M \psi_i(x). \tag{29}$$

By setting

$$\mathcal{W}_i^l(t) = \langle w(t,x), \psi_i(x)\rangle_M, \tag{30}$$

we have the Galerkin form of the projection to the POD space, which is

$$w(t,x) = \sum_{i=1}^{l} \mathcal{W}_i^l(t) \psi_i(x), \tag{31}$$

where the Fourier coefficients $\mathcal{W}_i^l$, $1 \le i \le l$, are functions mapping $[0,T]$ onto $\mathbb{R}$. Since for $l = m$, we have $w(t,x) = w^l(t,x)$, it can be deduced that $w^l(t,x)$ gives an approximation of $w(t,x)$ provided that $l \le m$.

We recall the weak solution approach described earlier for Burgers' equation that led to

$$\begin{aligned}&\int_0^L \left(w_t(t,x) + \frac{1}{2}[w^2(t,x)]_x \right) v(x)\,\mathrm{d}x \\ &- \nu \left[u_2(t)v(L) - u_1(t)v(0) - \int_0^L w_x(t,x)v'(x)\,\mathrm{d}x \right] \\ &\qquad = \int_0^L f(t,x)v(x)\,\mathrm{d}x.\end{aligned} \tag{32}$$

Since $v(x)$ is arbitrary and piecewise smooth, we let $v(x) = \psi_j(x), j = 1, 2, ..., l$ and use POD Galerkin projection that results in

$$\begin{aligned}&\sum_{i=1}^{l} \dot{\mathcal{W}}_i^l(t) \int_0^L \psi_i(x)\psi_j(x)\,\mathrm{d}x \\ &\quad + \frac{1}{2} \int_0^L \left(\left[\sum_{i=1}^{l} \mathcal{W}_i^l(t)\psi_i(x) \right]^2 \right)_x \psi_j(x)\,\mathrm{d}x \\ &\quad - \nu \left[u_2(t)\psi_j(L) - u_1(t)\psi_j(0) \right] \\ &+ \nu \sum_{i=1}^{l} \mathcal{W}_i^l(t) \int_0^L \psi_i'(x)\psi_j'(x)\,\mathrm{d}x = \int_0^L f(t,x)\psi_j(x)\,\mathrm{d}x.\end{aligned}$$

Considering the following notation

$$\begin{aligned}M_{ij}^l = \big\langle \psi_i(x), \psi_j(x) \big\rangle,\ S_{ij}^l = \big\langle \psi_i'(x), \psi_j'(x) \big\rangle, \\ F_j^l(t) = \big\langle f(t,x), \psi_j(x) \big\rangle, \\ (N^l(\mathcal{W}^l(t)))_j = \frac{1}{2} \int_0^L N\big(w(t,x)\big)\psi_j(x)\,\mathrm{d}x,\end{aligned}$$

reduced order model becomes

$$M^l \dot{\mathcal{W}}^l(t) + \nu S^l \mathcal{W}^l(t) + N^l(\mathcal{W}^l(t)) - \nu L^l U = F^l(t). \tag{33}$$

It is noted that when the basis functions are orthonormal, $M^l = I_r$. Matrix S^l in the reduced-order model can also be obtained from the original full-order matrices by expanding the POD basis functions as

$$\begin{aligned}S_{ij}^l &= \big\langle \psi_i'(x), \psi_j'(x) \big\rangle = \int_0^L \psi_i'(x)\psi_j'(x)\,\mathrm{d}x \\ &= \int_0^L \sum_{k=0}^{N} \Psi_{ki} \mathcal{N}_k'(x) \sum_{m=0}^{N} \Psi_{mj} \mathcal{N}_m'(x)\,\mathrm{d}x \\ &= \sum_{k=0}^{N} \sum_{m=0}^{N} \Psi_{ki}\Psi_{mj} \int_0^L \mathcal{N}_k'(x)\mathcal{N}_m'(x)\,\mathrm{d}x.\end{aligned} \tag{34}$$

This can be written in the following matrix form

$$S^l = (\Psi^l)^\top S \Psi^l. \tag{35}$$

Hence, we have characterized matrix S^l in terms of full order matrix S and the POD basis functions. Same procedure can be implemented to represent the nonlinear term in the reduced-order model in terms of the full order matrices. To archive this, we first need to find the relationship between the coefficients $\mathcal{W}_1(t), \dots, \mathcal{W}_N(t)$ and $\mathcal{W}_1^l(t), \dots, \mathcal{W}_l^l(t)$. Considering the fact that the solution w can be expressed in either l-dimensional reduced-order space or $N+1$-dimensional full order system, we have

$$w(t,x) \approx \sum_{i=1}^{l} \mathcal{W}_i^l(t)\psi_i(x) \approx \sum_{i=0}^{N} \mathcal{W}_i(t)\mathcal{N}_i(x). \tag{36}$$

As described earlier, the POD basis functions can be written as a linear combination of the FE basis functions, and hence

$$\sum_{i=0}^{N} \mathcal{W}_i(t)\mathcal{N}_i(x) \approx \sum_{i=1}^{l} \mathcal{W}_i^l(t) \sum_{m=0}^{N} \Psi_{mi}\mathcal{N}_m'(x). \tag{37}$$

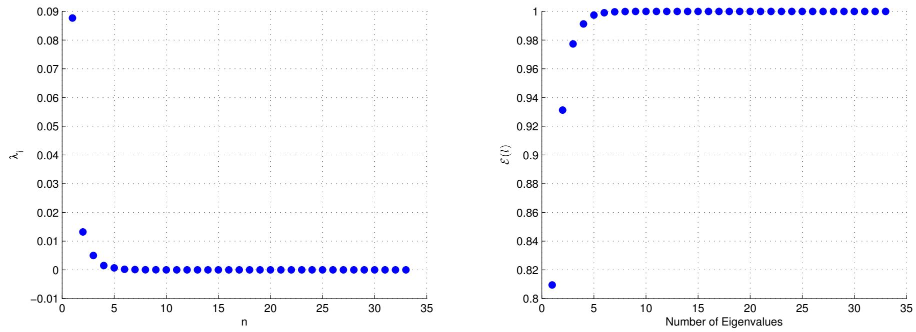

Fig. 1. The extracted eigenvalues corresponding to the POD eigenvectors (left); The energy captured by different chosen number of eigenvalues (right).

This describes the Fourier coefficients in the compact form as

$$\mathcal{W}(t) \approx \Psi^l \mathcal{W}^l(t). \quad (38)$$

Substituting (38) in (12) and using Hadamard product, we obtain

$$N^l(\mathcal{W}^l(t)) = \frac{1}{2}(\Psi^l)^\top K(\Psi^l \mathcal{W}^l(t)) \circ (\Psi^l \mathcal{W}^l(t)). \quad (39)$$

We further define

$$F^l(t) = (\Psi^l)^\top F(t), \;\; L^l = (\Psi^l)^\top L, \quad (40)$$

and hence the reduced order model is described by

$$\dot{\mathcal{W}}^l(t) + \nu S^l \mathcal{W}^l(t) + \frac{1}{2}(\Psi^l)^\top K(\Psi^l \mathcal{W}^l(t)) \circ (\Psi^l \mathcal{W}^l(t)) - \nu L^l U = F^l(t). \quad (41)$$

State-space Representation

After deriving the reduced-order model for the Burgers' PDE in the form of $N+1$ ordinary differential equations in (13) using finite element method, we can describe the system of ODEs in the state-space form with $N+1$ states as

$$\dot{\mathcal{W}}(t) = A\mathcal{W}(t) + \mathbf{h}(t, \mathcal{W}(t), U(t)), \quad (42)$$

where

$$A = -\nu M^{-1}S, \; \mathbf{h}(t, \mathcal{W}(t), U(t)) = -\frac{1}{2}M^{-1}K\mathcal{W}^2(t) + M^{-1}F(t) + \nu M^{-1}LU(t).$$

The reduced order state-space model is then represented by

$$\dot{\mathcal{W}}^l(t) = A^l \mathcal{W}^l(t) + \mathbf{g}(t, \mathcal{W}^l(t), U(t)), \quad (43)$$

where

$$A^l = -\nu S^l, \; \mathbf{g}(t, \mathcal{W}^l(t), U(t)) = -\frac{1}{2}(\Psi^l)^\top K(\Psi^l \mathcal{W}^l(t)) \circ (\Psi^l \mathcal{W}^l(t)) + F^l(t) + \nu L^l U(t).$$

VI. Simulation Results and Discussions

In order to assess the performance of the presented model reduction method, an example of a viscous Burgers' equation is discussed in this section. Our goal is to determine whether the reduced-order model can accurately estimate the full-order FE model. In this example, the forcing term in (1) is considered to be zero that translates to the so-called viscous Burgers' equation. The initial condition is also assumed to be

$$w_0(x) = \begin{cases} 100(sin(8\pi x) - 2x), & \text{if } x \in (0, \frac{1}{4}] \\ 0, & \text{otherwise.} \end{cases} \quad (44)$$

It is also assumed that the boundary conditions, i.e., inputs to the state-space models, are sinusoidal functions as

$$u_1(t) = 0.8sin(3t), u_2(t) = 0.5sin(3t). \quad (45)$$

As the first step, the eigenvalues corresponding to the POD method are extracted. Figure 1 shows the eigenvalues in a descending order. Also, the percentage of the total energy captured by the chosen number of eigenvalues is shown in Figure 1. In order to investigate the accuracy of the implemented model order reduction approach, FEM and 7-dimensional POD solution of Burgers' equation is shown in Figure 2. According to Figure 1, the first 7 eigenvalues capture more than 99% of the system's total energy. This is observed in Figure 2, where the POD solution closely matches the FEM solution.

The reduced and full-order open-loop models are simulated for a given viscosity $\nu = 0.01$ (or *Re* = 100) to gauge the performance of the model reduction approach for a class of physical flows. Figures 3 illustrates the output of the FE and reduced order POD models for a sinusoidal input signal in the given example. It is observed that by increasing the number of basis functions to 7, we can achieve a very close match between the output of both models. Finally, to quantify the model accuracy, we consider the *Best Fit Rate* (BFR) defined as

$$\text{BFR} = \max\left(1 - \frac{\|y_i(k) - \hat{y}_i(k)\|_{l_2}}{\|y_i(k) - \bar{y}_i\|_{l_2}}, 0\right), \quad (46)$$

where y_i and $\hat{y}_i$ represent the i^{th} output of the FEM and POD state-space models, respectively, and $\bar{y}_i$ is the mean value of the i^{th} output of the FEM model. In fact, the obtained BFR measures the matching between the output of the reduced order models and the output of the FE model. Table I shows the BFR of the reduced order models. Also shown in Table I is the mean-squared error (MSE) between the reduced-order model output and the original FE model output.

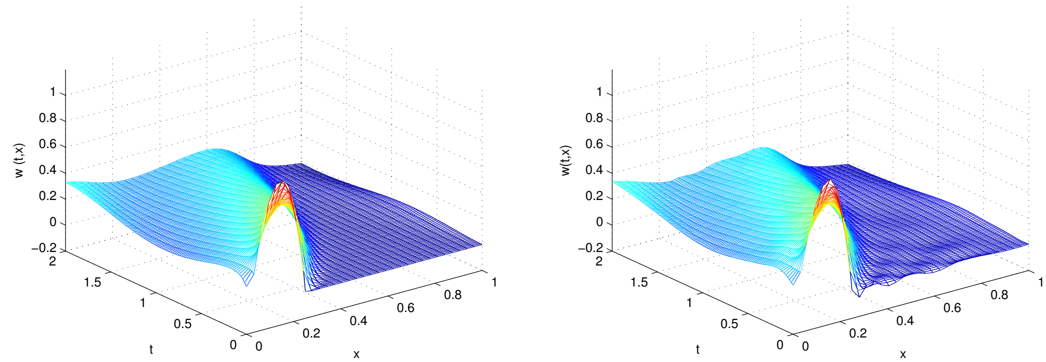

Fig. 2. Finite element solution for 32 spatial points (left), and POD solution with 7 basis functions (right) both with $\nu = 0.01$.

TABLE I
THE MSE AND BFR OF THE OUTPUT SIGNAL OF THE REDUCED ORDER MODELS WITH SINUSOIDAL INPUTS

	POD (3 Bases)		POD (5 Bases)		POD (7 Bases)	
Output	MSE	BFR	MSE	BFR	MSE	BFR
y_1	0.0021	0.0886	2.1836e-04	0.7042	1.3275e-05	0.9271
y_2	1.0203e-04	0.4909	6.7584e-06	0.8690	4.2925e-06	0.8956

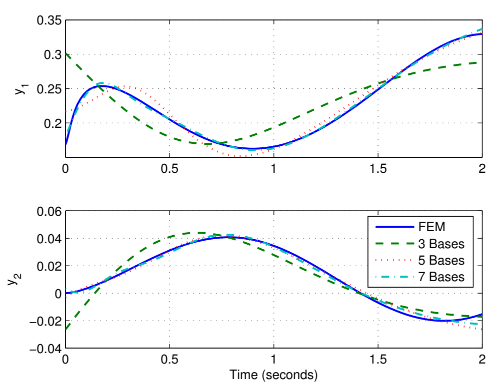

Fig. 3. The output of the FE model and the reduced-order model obtained using POD for different number of basis functions.

VII. Concluding Remarks

In this paper, we presented results on developing a reduced-order state-space model for a nonlinear parabolic PDE to overcome several drawbacks arising from implementing the full-order model specially for real-time applications. The model obtained used POD Galerkin method, and guaranteed a highly accurate approximation of the original model. We showed that the combination of POD and the weak solution approach can lead to an accurate reduced-order model. The results demonstrated that increasing the number of chosen eigenvalues would represent the original system with a higher accuracy. We are currently examining the design of robust nonlinear controllers based on the reduced-order nonlinear model derived in this paper.

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