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Sampled-data Control of Linear Parameter Varying Time-delay Systems Using State Feedback

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Abstract— **A sampled-data system includes the combination of continuous-time and discrete-time signals within the framework of hybrid dynamical systems. In this paper, we address the sampled-data control design problem for linear parameter varying (LPV) systems with state delay. The design objective is to find a discrete-time controller to ensure the closed-loop system stability and a prescribed level of performance, which in this study is defined to be the $\mathscr{H}_\infty$ norm of the closed-loop system. Presence of the state delay in the system dynamics, as well as the hybrid structure of the closed-loop system due to the augmentation of the continuous-time plant and discrete-time controller calls for a new control strategy. To this aim, we utilize one of the most recently developed and less conservative approaches in designing a gain-scheduled controller for state-delay LPV systems to formulate the design problem in terms of a set of linear matrix inequality (LMI) conditions. To this end, we utilize the *input delay* approach to map the closed-loop hybrid system into a continuous-time one by introducing a new delay term in the original state delay plant. The effectiveness of the proposed method is examined through numerical simulation on a mechanical system.**

I. INTRODUCTION

An inordinate number of practical systems exist in continuous-time domain, for which it is more convenient to design a controller in the same domain. However, the implementation of the designed controller has to be often carried out in a digital device. To this aim, a designed continuous-time controller can be simply discretized using conventional methods, *e.g.*, trapezoidal approximation. In this approach, known as the indirect method, the effect of sampling frequency does not come to play until the very last step, where the controller is discretized. Therefore, the need for redesigning the controller for different sampling rates is bypassed [1]. But since this method disregards the effect of sampling and holding devices in the design process, increasing the sampling period degrades the effectiveness of the control action, which may result in the instability of the closed-loop system. Conversely, the so-called direct sampled-data design always takes into account the effect of converter devices in the design process [3]. The main essence of all direct methods is to map the hybrid closed-loop system to either discrete-time or continuous-time domain and then using an existing design method in the associated time domain. In this study, we are interested in the latter case, by employing the input-delay method proposed in [4] and further developed in [5], [10]. This method turns a piecewise constant signal $u_d(t_k)$ that changes at some sampling instants t_k and remains constant between two consecutive instants ($t_k \le t < t_{k+1}$) into a continuous-time signal $u(t)$ by introducing a time-varying delay as follows

$$u(t) = u_d(t_k) = u_d(t-(t-t_k)) = u_d(t-\tau_k(t)), \quad t_k \le t < t_{k+1} \qquad (1)$$

where $\tau_k(t)$ represents a fast-varying delay ($\dot{\tau}_k = 1$). Utilizing this technique in the sampled-data control problem, we can deal with a pure continuous-time system including delay.

The main focus of the present work is to examine the sampled-data control design for linear parameter varying (LPV) systems with state delay. LPV systems include a class of linear systems, whose dynamics depends on time-varying parameters, also referred to as scheduling parameters, which are assumed to be known in real-time. In this study, we propose a method for the design of sampled-data controllers for time delay LPV systems using a parameter-dependent state feedback law. The proposed design method guarantees asymptotic stability and optimized energy-to-energy gain of the closed-loop system from disturbance input to the system output.

The literature of stability analysis and control of time delay systems is reach (the interested reader is referred to [6], [8] and numerous references therein). Also, there has been recent efforts devoted to the control design problem for time delay LPV systems [2], [11]. The present work is mainly inspired by the authors' recent work in [14], where the authors developed a set of linear matrix inequality (LMI)-based conditions for the analysis and synthesis of state-delay LPV systems, which were less conservative compared to the existing work in the literature. In this work, we use a parameter-dependent Lyapunov-Krasovskii functional leading to a delay-dependent synthesis method that can handle fast-varying time delays. We note that the fast-varying delays are encountered in our work due to the use of input delay technique. In our previous related work ([7]), we examined the sampled-data control design for LPV systems without intrinsic delay.

The notation used in this paper is standard. $\mathbb{R}$ denotes the set of real numbers. $\mathbb{R}^n$ and $\mathbb{R}^{k\times m}$ are used to denote the set of real vectors of dimension n and the set of real $k\times m$ matrices, respectively. In addition, $\mathbb{S}^{n\times n}$ and $\mathbb{S}_+^{n\times n}$ denote the set of real symmetric $n\times n$ matrices and symmetric positive definite matrices, respectively. In a symmetric matrix, the asterisk $\star$ in the (i,j) element denotes transpose of the (j,i) element.

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II. Problem statement

We consider the following state-space representation for a state-delay linear parameter varying (LPV) system

$$\dot{x}(t) = A(\rho(t))x(t) + A_h(\rho(t))x(t-h(t)) + B_1(\rho(t))w(t) + B_2(\rho(t))u(t)$$
$$z(t) = C(\rho(t))x(t) + C_h(\rho(t))x(t-h(t)) + D_1(\rho(t))w(t) + D_2(\rho(t))u(t),$$

where $x(t) \in \mathbb{R}^n$ is the state vector that is assumed to be measurable, $z(t) \in \mathbb{R}^{n_z}$ is the vector of controlled outputs, $w(t) \in \mathbb{R}^{n_w}$ is the exogenous disturbance vector with finite energy containing process noise and $u(t) \in \mathbb{R}^{n_u}$ is the control input vector. The system matrices $A(\cdot)$, $A_h(\cdot)$, $B_1(\cdot)$, $B_2(\cdot)$, $C(\cdot)$, $C_h(\cdot)$, $D_1(\cdot)$ and $D_2(\cdot)$ are real-valued continuous functions of a time varying parameter vector $\rho(t)$ known as the scheduling parameter vector and of appropriate dimensions. The set of all allowable parameter trajectories is defined by

$$\mathscr{F}^{\nu}_{\mathscr{P}} \equiv \{\rho(t) \in \mathscr{P}, |\dot{\rho}_i(t)| \le \nu_i \quad i = 1,2,...,s\},$$

where $\mathscr{P}$ is a compact set of $\mathbb{R}^s$. Also $0 \le h(t) \le h_m$ is a scalar function denoting a time-varying delay with bounded variation. One may also assume that $h(t)$ depends on the scheduling parameters, *i.e.*, $h(t) = h(\rho(\cdot))$. To find the unique integral solution to (2), the initial condition of the state vector is considered to be $x(t) = \phi(t)$ for $t \in [-h_m, 0]$. In this study, we are interested in designing a sampled-data state-feedback controller that uses the discrete samples of system states as input and provides control input to the plant by holding the controller's discrete output. Illustrated in Fig. 1 is the closed-loop system configuration. We further assume

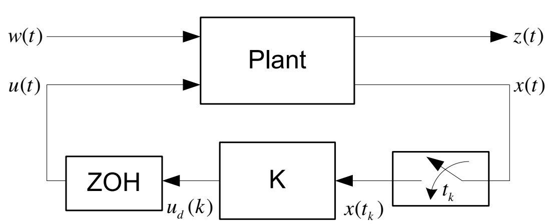

Fig. 1. Sampled data control of a continuous-time system

that the sampling instants are not necessarily equi-spaced but are constrained to $t_{k+1} - t_k \le \tau_m$ for two consecutive samples k and $k+1$. To potentially improve the performance of the closed-loop system, the control gain is assumed to be updated based on the value of the scheduling parameter at corresponding sampling instant, *i.e.*, $u_d(t_k) = K(\rho(t_k))x(t_k)$. Using a zero-order hold (ZOH shown in Fig. 1), the controller output becomes a piecewise-constant signal and is fed to the continuous-time plant, *i.e.*

$$u(t) = u_d(t_k) \quad t_k \le t < t_{k+1}. \qquad (2)$$

The design has to ensure the stability of the closed-loop hybrid system and leads to a sub-optimal performance measured in terms of the $\mathscr{H}_\infty$ norm of the closed-loop system. The γ-suboptimal $\mathscr{H}_\infty$ control problem seeks for a controller that yields $\|z\|_{\mathscr{L}_2} < \gamma \|w\|_{\mathscr{L}_2}$ for some positive scalar value γ. Due to the hybrid nature of the closed-loop system determined from the aforementioned sampled-data control problem, representation of the closed-loop system in a unified domain is difficult. In this paper, we use the *input delay* approach based on (1) to represent the digital control law as a continuous time-varying delay. The time-varying delay is bounded as $\tau_k \le t_{k+1} - t_k \le \tau_m$ and is also piecewise linear with the rate of variation $\frac{d\tau_k}{dt} = 1$ for $t \ne t_k$. The closed-loop interconnection of (2) and (2) is

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_h x(t-h) + A_\tau x(t-\tau_k) + B_1 w(t) \\ z(t) &= Cx(t) + C_h x(t-h) + C_\tau x(t-\tau_k) + D_1 w(t), \end{aligned} \qquad (3)$$

with

$$A_\tau(\rho(t)) = B_2(\rho(t)K(\rho(t_k)), \quad C_\tau(\rho(t)) = D_2(\rho(t)K(\rho(t_k)). \qquad (4)$$

For the sake of brevity, the dependency of the system matrices on ρ may be dropped throughout the paper. Next, we provide some useful lemmas.

Lemma 1 [13] (Cauchy-Schwartz Inequality): For any positive scalar h, positive definite matrix P and any $\nu(\alpha) \in \mathbb{R}^n$, we have

$$h\int_{t-h}^{t} \nu^T(\alpha)P\nu(\alpha)d\alpha \ge \left[\int_{t-h}^{t}\nu(\alpha)d\alpha\right]^T P \left[\int_{t-h}^{t}\nu(\alpha)d\alpha\right].$$

Lemma 2 [9] (Projection Lemma): Given a symmetric matrix Ψ and two matrices Λ and Γ of appropriate dimensions, the linear matrix inequality

$$\Psi + \Lambda^T\Theta^T\Gamma + \Gamma^T\Theta\Lambda < 0 \qquad (5)$$

has a feasible solution in terms of Θ if and only if

$$\mathscr{N}_\Lambda^T \Psi \mathscr{N}_\Lambda < 0, \qquad (6)$$

$$\mathscr{N}_\Gamma^T \Psi \mathscr{N}_\Gamma < 0, \qquad (7)$$

where $\mathscr{N}_\Lambda$ and $\mathscr{N}_\Gamma$ are any basis of the null-space of Λ and Γ, respectively.

III. Stability and Performance Analysis of LPV Systems With Time Delay

In this section, we present stability and $\mathscr{H}_\infty$ norm performance analysis conditions for time-delay LPV systems by deriving a set of linear matrix inequality (LMI) conditions. We note that the existing LMI-based results in the literature on analysis of time-delay LPV systems are only applicable to the systems, in which the rate of the time-delay is less than one. For the sampled-data control problem under study in this paper, we have $\dot{\tau} = 1$ and hence a set of new conditions must be derived.

A. Stability Analysis

We first consider the following unforced LPV system with two delay terms

$$\dot{x}(t) = A(\rho(t))x(t) + A_h(\rho(t))x(t-h) + A_\tau(\rho(t))x(t-\tau_k). \qquad (8)$$

Lyapunov-Krasovskii stability theory serves as a useful tool to achieve delay-dependent conditions for the stability analysis of the system represented by (8). To this aim, we need to find a positive definite functional with an infinitesimal upper bound, whose time derivative is negative. The interested reader is referred to [13], [6]. As the first result of this paper, we present the following theorem as a sufficient condition to ensure asymptotic stability of the LPV system represented by (8).

Theorem 1: The time-delay LPV system (8) is asymptotically stable for all $0 < h(t) \leq h_m$ with $\tau_k(t) \leq \tau_m$ and $\dot{\tau}_k = 1$, if there exist a continuously differentiable matrix function $P : \mathbb{R}^s \to \mathbb{S}_+^{n\times n}$, constant matrices $Q_h, R_h, R_\tau \in \mathbb{S}_+^{n\times n}$ such that for all $\rho(t) \in \mathscr{F}_{\mathscr{P}}^{v}$, there is a feasible solution to the following LMI problem

$$\begin{bmatrix} \Sigma_{1,1} & PA_h+R_h & PA_\tau+R_\tau & h_mA^TR_h & \tau_mA^TR_\tau \\ \star & -(1-\dot{h})Q_h-R_h & 0 & h_mA_h^TR_h & \tau_mA_h^TR_\tau \\ \star & \star & -R_\tau & h_mA_\tau^TR_h & \tau_mA_\tau^TR_\tau \\ \star & \star & \star & -R_h & 0 \\ \star & \star & \star & \star & -R_\tau \end{bmatrix} < 0, \quad (9)$$

with $\Sigma_{1,1} = A^TP + PA + \dot{P} + Q_h - R_h - R_\tau$.

Proof: We consider the following Lyapunov-Krasovskii functional

$$V(x_t,\rho) = V_1(x,\rho) + V_h(x_t,\rho) + V_\tau(x_t,\rho), \quad (10)$$

with

$$\begin{aligned} V_1(x,\rho) &= x^T(t)P(\rho(t))x(t) \\ V_h(x_t,\rho) &= \int_{t-h(t)}^{t} x^T(\xi)Q_hx(\xi)d\xi \\ &+ \int_{-h_m}^{0}\int_{t+\theta}^{t} \dot{x}^T(\xi)h_mR_h\dot{x}(\xi)\,d\xi d\theta \\ V_\tau(x_t,\rho) &= \int_{-\tau_m}^{0}\int_{t+\theta}^{t} \dot{x}^T(\xi)\tau_mR_\tau\dot{x}(\xi)\,d\xi d\theta, \end{aligned}$$

where $x_t(\theta)$ is used to represent $x(t+\theta)$ for $\theta \in [-h_m, 0]$. It can be shown that (10) is positive definite with infinitesimal upper bound functional. It is noted that (10) is chosen to be dependent on the LPV parameter vector $\rho(t)$ and the maximum value of delays to result in less conservative stability conditions [2]. In order for the system (8) to be asymptotically stable, it suffices that the time derivative of (10) along the system trajectory (8) is negative definite. We have

$$\dot{V}_1(x_t,\rho) = \dot{x}^TPx + x^TP\dot{x} + x^T\dot{P}x$$

and

$$\begin{aligned} \dot{V}_h &= x^T(t)Q_hx(t) - (1-\dot{h}(t))x^T(t-h)Q_hx(t-h) \\ &+ h_m^2\,\dot{x}^T(t)R_h\dot{x}(t) - \int_{t-h_m}^{t} \dot{x}^T(\theta)h_mR_h\dot{x}(\theta)d\theta. \end{aligned} \quad (11)$$

Since $h(t) \leq h_m$, the integral term in (11) satisfies

$$-\int_{t-h_m}^{t} \dot{x}^T(\theta)h_mR_h\dot{x}(\theta)\,d\theta \leq -\int_{t-h(t)}^{t} \dot{x}^T(\theta)h_mR_h\dot{x}(\theta)d\theta.$$

Employing Lemma 1, we can bound the right hand side of the above inequality by

$$\begin{aligned} &-\frac{h_m}{h(t)}\left(\int_{t-h(t)}^{t} \dot{x}^T(\theta)\,d\theta\right)^T R_h \left(\int_{t-h(t)}^{t} \dot{x}^T(\theta)\,d\theta\right) \\ &= -\frac{h_m}{h(t)}[x(t)-x(t-h)]^T R_h [x(t)-x(t-h)] \\ &\leq -[x(t)-x(t-h)]^T R_h [x(t)-x(t-h)] \end{aligned}$$

as a result of the fact that $-\frac{h_m}{h(t)} \leq -1$. Thus

$$\begin{aligned} \dot{V}_h &\leq x^TQ_hx - (1-\dot{h})x^T(t-h)Q_hx(t-h) \\ &+ h_m^2\,\dot{x}^TR_h\,\dot{x} - [x(t)-x(t-h)]^T R_h [x(t)-x(t-h)]. \end{aligned}$$

Similarly for V_τ, we obtain

$$\dot{V}_\tau \leq \tau_m^2\,\dot{x}^TR_\tau\,\dot{x} - [x(t)-x(t-\tau_k)]^T R_\tau [x(t)-x(t-\tau_k)].$$

Substituting in (10) for $\dot{V}_1$, $\dot{V}_h$ and $\dot{V}_\tau$, we obtain

$$\begin{aligned} \dot{V}(x_t,\rho) &\leq \dot{x}^TPx + x^TP\dot{x} + x^T\dot{P}x \\ &+ x^T\,Q_h\,x - (1-\dot{h})x^T(t-h)\,Q_h\,x(t-h) \\ &+ h_m^2\,\dot{x}^TR_h\,\dot{x} - [x(t)-x(t-h)]^T R_h [x(t)-x(t-h)] \\ &+ \tau_m^2\,\dot{x}^TR_\tau\,\dot{x} - [x(t)-x(t-\tau_K)]^T R_\tau [x(t)-x(t-\tau_k)]. \end{aligned} \quad (12)$$

Up to this point, we have found an inequality representing the stability condition. To derive an inequality condition in a matrix form, we replace for $\dot{x}$ in (12) from (8) and gather the relevant terms as follows

$$\begin{aligned} \dot{V}(x_t,\rho) \leq \begin{bmatrix} x(t) \\ x(t-h) \\ x(t-\tau_k) \end{bmatrix}^T \Bigg(\mathscr{X} + \begin{bmatrix} h_mA^TR_h \\ h_mA_h^TR_h \\ h_mA_\tau^TR_h \end{bmatrix} R_h^{-1} \begin{bmatrix} h_mA^TR_h \\ h_mA_h^TR_h \\ h_mA_\tau^TR_h \end{bmatrix}^T \\ + \begin{bmatrix} \tau_mA^TR_\tau \\ \tau_mA_h^TR_\tau \\ \tau_mA_\tau^TR_\tau \end{bmatrix} R_\tau^{-1} \begin{bmatrix} \tau_mA^TR_\tau \\ \tau_mA_h^TR_\tau \\ \tau_mA_\tau^TR_\tau \end{bmatrix}^T \Bigg) \begin{bmatrix} x(t) \\ x(t-h) \\ x(t-\tau_k) \end{bmatrix}, \end{aligned} \quad (13)$$

where

$$\mathscr{X} = \begin{bmatrix} \Sigma_{1,1} & PA_h+R_h & PA_\tau+R_\tau \\ \star & -(1-\dot{h})Q_h-R_h & 0 \\ \star & \star & -R_\tau \end{bmatrix}.$$

Using (13) and applying the Schur complement twice, we assure $\dot{V}(x_t,\rho) < 0$ the condition (9) is attained and this completes the proof.

Remark 1: In the matrix inequality (9), the (1,1) entry contains the derivative term $\dot{P}$ that can be replaced by $\frac{\partial P}{\partial \rho}\dot{\rho}$. Due to the affine dependency of this matrix inequality on $\dot{\rho}$, it is only required to solve the feasibility problem at vertices of $\dot{\rho}$. Therefore, in matrix inequality (9), $\dot{P}$ can be replaced with $\sum_{i=1}^{s} \pm \left(v_i \frac{\partial P}{\partial \rho_i}\right)$. The summation means that every combination of + and - should be included in the inequality. That is, the inequality actually represents 2^s different combinations. In case the delay term h is dependent on the LPV parameter vector $\rho(t)$, the derivative term $\dot{h}$ can be replaced by $\sum_{i=1}^{s} \pm \left(v_i \frac{\partial h}{\partial \rho_i}\right)$; otherwise, the term $\dot{h}$ is simply replaced by its corresponding upper and lower bounds.

B. Performance Analysis

Next, we consider the state-space model (3) and derive the corresponding performance analysis condition.

Theorem 2: The LPV system represented by (3) is asymptotically stable and the condition $\|z\|_{\mathscr{L}_2} \leq \gamma\|w\|_{\mathscr{L}_2}$ is satisfied for all $0 < h(t) \leq h_m$ with $\tau_k(t) \leq \tau_m$, $\dot{\tau}_k = 1$ and zero initial condition if there exist a continuously differentiable matrix function $P:\mathbb{R}^s \to \mathbb{S}_+^{n\times n}$, constant matrices $Q_h, R_h, R_\tau \in \mathbb{S}_+^{n\times n}$ and a positive scalar γ such that for any admissible parameter trajectory $\rho(t) \in \mathscr{F}_{\mathscr{P}}^{v}$, the following matrix inequality condition holds true

$$\begin{bmatrix} \Sigma_{1,1} & PA_h+R_h & PA_\tau+R_\tau & PB_1 & C^T & h_mA^TR_h & \tau_mA^TR_\tau \\ \star & \Sigma_{2,2} & 0 & 0 & C_h^T & h_mA_h^TR_h & \tau_mA_h^TR_\tau \\ \star & \star & -R_\tau & 0 & C_\tau^T & h_mA_\tau^TR_h & \tau_mA_\tau^TR_\tau \\ \star & \star & \star & -\gamma I & D_1^T & h_mB_1^TR_h & \tau_mB_1^TR_\tau \\ \star & \star & \star & \star & -\gamma I & 0 & 0 \\ \star & \star & \star & \star & \star & -R_h & 0 \\ \star & \star & \star & \star & \star & \star & -R_\tau \end{bmatrix} < 0, \quad (14)$$

where $\Sigma_{2,2} = -(1-\dot{h})Q_h - R_h$.

Proof: We first consider a Lyapunov-Krasovskii functional similar to the one introduced in Theorem 1. Next, we apply the congruent transformation $\mathscr{T}$ to inequality (14), where

$$\mathscr{T} = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I \\ 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 \end{bmatrix}.$$

In the obtained inequality, it can be observed that the negative definiteness of the upper left 4×4 block matrix, in light of Theorem 1, concludes the asymptotical stability of the system (3). Applying Schur complement to (14) three times results in

$$\begin{bmatrix} \Sigma_{1,1} & PA_h+R_h & PA_\tau+R_\tau & PB_1 \\ \star & \Sigma_{2,2} & 0 & 0 \\ \star & \star & -R_\tau & 0 \\ \star & \star & \star & -\gamma I \end{bmatrix} + \begin{bmatrix} C^T \\ C_h^T \\ C_\tau^T \\ D_1^T \end{bmatrix} \gamma^{-1} \begin{bmatrix} C^T \\ C_h^T \\ C_\tau^T \\ D_1^T \end{bmatrix}^T + \begin{bmatrix} A^T \\ A_h^T \\ A_\tau^T \\ B_1^T \end{bmatrix} h_m^2 R_h \begin{bmatrix} A^T \\ A_h^T \\ A_\tau^T \\ B_1^T \end{bmatrix}^T + \begin{bmatrix} A^T \\ A_h^T \\ A_\tau^T \\ B_1^T \end{bmatrix} \tau_m^2 R_\tau \begin{bmatrix} A^T \\ A_h^T \\ A_\tau^T \\ B_1^T \end{bmatrix}^T < 0.$$

Multiplying this inequality from left and right by $\zeta^T(t)$ and $\zeta(t)$, respectively, where $\zeta(t) = [x^T(t)\ x^T(t-h)\ x^T(t-\tau_k)\ w^T(t)]^T$. With further algebraic manipulations, we obtain

$$\begin{aligned} &\dot{x}^TPx + x^TP\dot{x} + x^T\dot{P}x + x^TQ_hx - (1-\dot{h})x^T(t-h)Q_hx(t-h) \\ &+h_m^2\dot{x}^TR_h\dot{x} - [x(t)-x(t-h)]^T R_h [x(t)-x(t-h)] \\ &+\tau_m^2\dot{x}^TR_\tau\ \dot{x} - [x(t)-x(t-\tau_k)]^T R_\tau [x(t)-x(t-\tau_k)] \\ &-\gamma w^T(t)w(t) + \frac{1}{\gamma}z^T(t)z(t) < 0, \end{aligned}$$

and using (12), we have $\dot{V}(x_t,\rho) - \gamma w^T(t)w(t) + \frac{1}{\gamma}z^T(t)z(t) < 0$. Integrating both sides of this inequality from 0 to ∞ and using $V|_{t=0} = V|_{t=\infty} = 0$ (due to the asymptotical stability and zero initial condition), we arrive at $\|z\|_{\mathscr{L}_2} \le \gamma \|w\|_{\mathscr{L}_2}$.

IV. Introduction of Slack Variables

Substitution of system matrices of (3) in (14), results in a bilinear matrix inequality problem due to the byproduct of the controller matrix K with the unknown matrix function P and the matrices R_m and R_τ. Therefore, we will seek an alternative method based on the introduction of *slack variables* to reformulate the corresponding problem to ensure that an LMI condition is achieved.

Theorem 3: The LPV system represented by (3) is asymptotically stable for all $0 < h(t) \le h_m$ and $\tau_k(t) \le \tau_m$, $\dot{\tau}_k = 1$ and satisfies $\|z\|_{\mathscr{L}_2} \le \gamma\|w\|_{\mathscr{L}_2}$ if there exist a continuously differentiable matrix function P, constant matrices Q_h, R_h, R_τ, V_1, V_2, V_3, $V_4 \in \mathbb{S}_+^{n\times n}$ and a positive scalar γ such that for any admissible parameter trajectory $\rho(t) \in \mathscr{F}_{\mathscr{P}}^{v}$, the following LMI problem has a feasible solution

$$\begin{bmatrix} -V_1-V_1^T & P-V_2^T+V_1A & -V_3^T+V_1A_h & -V_4^T+V_1A_\tau & V_1B_1 & 0 & h_mR_h & \tau_mR_\tau \\ \star & \Sigma_{2,2} & R_h+A^TV_3^T+V_2A_h & R_\tau+A^TV_4^T+V_2A_\tau & V_2B_1 & C^T & 0 & 0 \\ \star & \star & -(1-\dot{h})Q_h-R_h+A_h^TV_3^T+V_3A_h & A_h^TV_4^T+V_3A_\tau & V_3B_1 & C_h^T & 0 & 0 \\ \star & \star & \star & -R_\tau+A_\tau^TV_4^T+V_4A_\tau & V_4B_1 & C_\tau^T & 0 & 0 \\ \star & \star & \star & \star & -\gamma I & D_1^T & 0 & 0 \\ \star & \star & \star & \star & \star & -\gamma I & 0 & 0 \\ \star & \star & \star & \star & \star & \star & -R_h & 0 \\ \star & \star & \star & \star & \star & \star & \star & -R_\tau \end{bmatrix} < 0, \quad (15)$$

where $\Sigma_{2,2} = \dot{P} + Q_h - R_h - R_\tau + A^TV_2^T + V_2A$.

Proof: We rewrite (15) as $\Psi + \Lambda^T\Theta^T\Gamma + \Gamma^T\Theta\Lambda < 0$ with

$$\Psi = \begin{bmatrix} 0 & P & 0 & 0 & 0 & 0 & h_mR_h & \tau_mR_\tau \\ \star & \Delta_{2,2} & R_h & R_\tau & 0 & C^T & 0 & 0 \\ \star & \star & \Delta_{3,3} & 0 & 0 & C_h^T & 0 & 0 \\ \star & \star & \star & -R_\tau & 0 & C_\tau^T & 0 & 0 \\ \star & \star & \star & \star & -\gamma I & D_1^T & 0 & 0 \\ \star & \star & \star & \star & \star & -\gamma I & 0 & 0 \\ \star & \star & \star & \star & \star & \star & -R_h & 0 \\ \star & \star & \star & \star & \star & \star & 0 & -R_\tau \end{bmatrix}, \quad (16)$$

where

$$\begin{aligned} \Delta_{2,2} &= \dot{P} + Q_h - R_h - R_\tau, \quad \Delta_{3,3} = -(1-\dot{h})Q_h - R_h, \\ \Lambda &= \begin{bmatrix} -I & A & A_h & A_\tau & B_1 & 0 & 0 & 0 \end{bmatrix}, \\ \Theta &= \begin{bmatrix} V_1^T & V_2^T & V_3^T & V_4^T \end{bmatrix}^T, \\ \Gamma &= \begin{bmatrix} diag(I,I,I,I) & diag(0,0,0,0) \end{bmatrix}. \end{aligned}$$

The matrix variables V_i $(i = 1,\ldots,4)$ are called slack variables [12]. Next, we use Lemma 2 (Projection Lemma) by finding the bases for the null space of Λ and Γ as

$$\mathscr{N}_\Lambda = \begin{bmatrix} A,\ A_h\ ,\ A_\tau\ ,\ B_1\ ,\ 0\ ,\ 0\ ,\ 0 \\ diag(\ I,\ I,\ I,\ I,\ I,\ I,\ I) \end{bmatrix}, \quad \mathscr{N}_\Gamma = \begin{bmatrix} diag(0,0,0,0) \\ diag(I,I,I,I) \end{bmatrix}.$$

Using the solvability condition (6) results in the LMI condition (14). On the other hand, the second solvability condition, *i.e.*, (7), leads to the following LMI

$$\begin{bmatrix} -\gamma I & D_1^T & 0 & 0 \\ \star & -\gamma I & 0 & 0 \\ \star & \star & -R_h & 0 \\ \star & \star & \star & -R_\tau \end{bmatrix} < 0, \quad (17)$$

which is part of the LMI (15) and is always satisfied. In summary, feasibility of the LMI condition (15) ensures that the LMI problem (14) is feasible and hence the proof of Theorem 3 is complete.

V. State-feedback $\mathscr{H}_\infty$ Sampled-data Control Design for Time-delay LPV Systems

Lemma 3: Consider the time-delay LPV system (3). There exists a sampled-data controller of the form (2) such that the corresponding hybrid closed-loop system is asymptotically stable and satisfies $\|z\|_{\mathscr{L}_2} \le \gamma \|w\|_{\mathscr{L}_2}$ for all $0 < h(t) \le h_m$ and $\tau_k(t) \le \tau_m,\ \dot{\tau}_k = 1$ if there exist a continuously differentiable matrix function $\tilde{P}: \mathbb{R}^s \to \mathbb{S}_+^{n\times n}$, matrix function $\tilde{K}: \mathbb{R}^s \to \mathbb{S}_+^{n_u\times n}$, constant matrices $\tilde{Q}_h, \tilde{R}_h, \tilde{R}_\tau$ and $U \in \mathbb{S}_+^{n\times n}$, a positive scalar γ and real scalars λ_2, λ_3 and λ_4 such that the LMI

$$\begin{bmatrix} -2U & \tilde{P}-\lambda_2 U + AU & -\lambda_3 U + A_h U & -\lambda_4 U + B_2\tilde{K} & B_1 & 0 & h_m\tilde{R}_h & \tau_m \tilde{R}_\tau \\ \star & \tilde{\Sigma}_{2,2} & \tilde{R}_h + \lambda_3 UA^T + \lambda_2 A_h U & \tilde{R}_\tau + \lambda_4 UA^T + \lambda_2 B_2 \tilde{K} & \lambda_2 B_1 & UC^T & 0 & 0 \\ \star & \star & -(1-\dot{h})\tilde{Q}_h - \tilde{R}_h + \lambda_3(UA_h^T + A_h U) & \lambda_4 UA_h^T + \lambda_3 B_2\tilde{K} & \lambda_3 B_1 & UC_h^T & 0 & 0 \\ \star & \star & \star & -\tilde{R}_\tau + \lambda_4(\tilde{K}^T B_2^T + B_2\tilde{K}) & \lambda_4 B_1 & \tilde{K}^T D_2^T & 0 & 0 \\ \star & \star & \star & \star & -\gamma I & D_1^T & 0 & 0 \\ \star & \star & \star & \star & \star & -\gamma I & 0 & 0 \\ \star & \star & \star & \star & \star & \star & -\tilde{R}_h & 0 \\ \star & \star & \star & \star & \star & \star & 0 & -\tilde{R}_\tau \end{bmatrix} < 0, \quad (18)$$

with $\tilde{\Sigma}_{2,2} = \dot{\tilde{P}} + \tilde{Q}_h - \tilde{R}_h - \tilde{R}_\tau + \lambda_2(UA^T + AU)$, is feasible for any admissible trajectory $\rho(t) \in \mathscr{F}_{\mathscr{P}}^{\nu}$. Then, the corresponding sampled-data controller gain in (2) is obtained from

$$K(\rho(t))|_{t=t_k} = \tilde{K}(\rho(t))|_{t=t_k} U^{-1}. \quad (19)$$

Proof: Starting from the LMI (15), we substitute the system matrices from (4). Next, we impose a constraint on the slack variables as $V_1 = V,\ V_i = \lambda_i V\ \ (i = 2,3,4)$, where λ_i's are real scalars. We note that the above choices are made to ensure that the resulting matrix inequality problem is linear. Next, applying the congruent transformation $T = \mathrm{diag}(U,U,U,U,I,I,U)$ with $U = V^{-1}$ and considering $U^T PU = \tilde{P}$, $U^T Q_h^T U = \tilde{Q}_h$, $U^T R_h U = \tilde{R}_h$, $U^T R_\tau U = \tilde{R}_\tau$ and $KU = \tilde{K}$, the LMI (18) is obtained and consequently (19) gives the controller gain. This completes the proof.

Remark 2: The LMI problem (18) is infinite-dimensional due to the dependency of the system matrices on LPV parameters continuously. A standard approach to solve the parameterized LMIs like (18) is to initially select some basis functions to represent the dependency of the matrix variables on the LPV parameters and then grid the parameter space. Finally, the obtained finite-dimensional LMI problem is solved at the grid points and then checked on a finer grid.

VI. Illustrative Example

As an illustrative example, we design a sampled-data controller to control the chattering of a milling machine, whose simplified mechanical model is depicted in Fig. 2. The dynamic model of this system can be formulated as an LPV system containing a parameter-dependent time delay [11], [13]. The system consists of a spindle of mass $m_2 = 2\ kg$

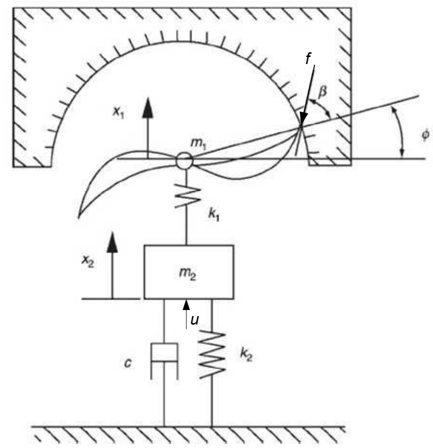

Fig. 2. A simplified schematic of milling process

and a two-blade cutter of mass $m_1 = 1\ kg$. Also two springs with stiffness $k_1 = 10\ N/m$ and $k_2 = 20\ N/m$ and a damping with the coefficient $c = 0.5\ N/m.s$ are lumped in the model. The dynamic equations of the system is

$$\ddot{x}_1 = \frac{1}{m_1}[-k_1 - k\sin(\phi)\sin(\phi+\beta)]x_1 + \frac{k_1}{m_1}x_1(t-\frac{\pi}{\omega}) + \frac{k_1}{m_1}x_2 + \frac{k_1}{m_1}w(t)$$
$$\ddot{x}_2 = \frac{k_1}{m_2}x_1 - \frac{k_1+k_2}{m_2}x_2 - \frac{c}{m_2}\dot{x}_2 + \frac{k_1}{m_2}u, \quad (20)$$

whit the assumption of $\beta = 70^\circ$. The obtained model relies on two measurable parameters ϕ and ω. First, we note that

$$\sin(\phi)\sin(\phi+\beta) = 0.5[\cos(\beta) - \cos(2\phi+\beta)] = 0.1710 - 0.5\cos(2\phi+\beta).$$

Next, we define the scheduling parameter vector as $\rho(t) = [\rho_1(t), \rho_2(t)]^T$ with $\rho_1(t) = \cos(2\phi(t)+\beta(t))$ and $\rho_2(t) = \omega(t)$. The rotation speed ω is assumed to vary between 200 rpm (20.94 rad/sec) and 2000 rpm (209.4 rad/sec), and maximum variation rate is assumed to be 500 rpm/sec (52.35 rad/sec^2). The parameter space associated with the LPV parameters is as follows

$$\rho_1(t) \in [-1 \quad 1], \qquad |\dot{\rho}_1| = |-2\omega\sin(2\phi(t)+\beta)| \le 418.9\ \text{(rad/sec)}$$
$$\rho_2(t) \in [20.94 \quad 209.4]\ \text{(rad/sec)}, \qquad |\dot{\rho}_2| = 104.7\ \text{(rad/sec}^2\text{)}.$$

Also for the given parameters, $0.015 < h(t) < 0.15$. Considering the state vector to be $x = [x_1\ x_2\ \dot{x}_1\ \dot{x}_2]^T$, the matrices associated with the model (3) of the system is

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -10.34+\rho_1(t) & 10 & 0 & 0 \\ 5 & -15 & 0 & -0.25 \end{bmatrix}, \quad A_h = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.34-\rho_1(t) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.5 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0 \\ 0 \\ 0.1 \end{bmatrix}, \tag{21}$$

where $z(t)$ is the fictitious system output reflecting the control design objectives. That implies that an $\mathscr{H}_\infty$ controller is sought to reduce the displacement of the elements, as well as penalizing large control actions. We assume that the system states are measurable in real-time. For effective control of the system, the choice of sampling frequency is crucial. Since two blades touch the surface twice per revolution, it is quite reasonable that sampling frequency be at least twice in a rotation. It is noted that since the angular velocity might vary during the milling process, the sampling period is not fixed and changes accordingly. In this example, we consider three samples per revolution, *i.e.*,

$$t_{k+1} = t_k + \frac{2\pi}{3\omega(t_k)}, \tag{22}$$

with $t_0 = 0$. Considering the bounds on the time delay, we have $h_m = 0.15$ and $\tau_m = \frac{2}{3}(0.15) = 0.1$ to be used in the LMI (15). In addition, we have to decide on the structure of the function variables $\tilde{P}$, $\tilde{Q}_h$ and $\tilde{K}$ involved in the LMI problem. In order to reduce the computational cost, we consider constant function variables (parameter-independent). It is noted that the optimal value of γ is quite sensitive to the value of the scalars λ_2, λ_3 and λ_4. These scalars can be optimized by performing a 3-dimensional search. For this example, our search resulted in $\lambda_2 = 1.3$, $\lambda_3 = 0.2$ and $\lambda_4 = 1.5$. Also the obtained $\mathscr{H}_\infty$ performance level was calculated to be $\gamma = 2.4$. Shown in Fig. 3 is the simulation results, indicating the displacement of the cutter x_1 and that of the spindle x_2 for a predefined test condition (with y-axis labeling on the left). It is assumed that the system is perturbed by a rectangular disturbance $w(t)$ of magnitude one over the time interval $t \in [0,4]$ and zero elsewhere. The blade rotational speed profile in round per minute (rpm) unit is chosen to be

$$\omega(t) = \begin{cases} 200 & 0 \le t < 2 \\ 800(t-2)+200 & 2 \le t < 3 \\ -800(t-7)+1000 & 7 \le t < 8 \\ 200 & 8 \le t \end{cases}$$

It is apparent that the proposed controller attenuates the disturbance successfully under the variable rotational speed. The control effort required for this study is shown in Fig. 3 (with y-axis labeling on the right), in which the different length of sampling periods is a result of using (22).

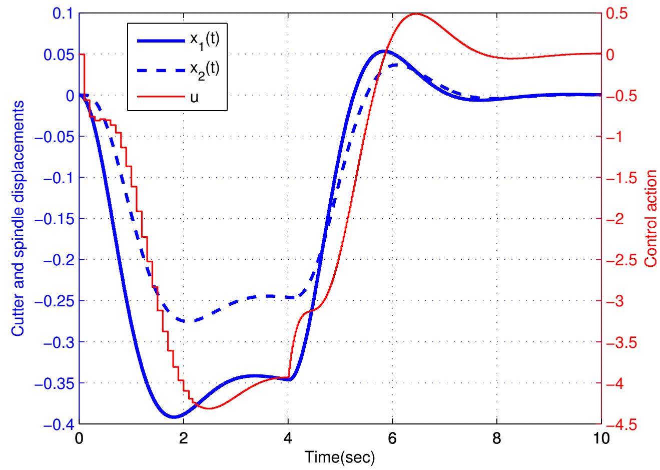

Fig. 3. Displacement of cutter and spindle

VII. Concluding Remarks

Sampled-data control design for a continuous-time system leads to a hybrid closed-loop system that is difficult to represent in a unified domain, either continuous-time or discrete-time. In this paper, we utilize the *input delay* approach for sampled-data control design of continuous-time state-delay LPV systems by mapping the hybrid closed-loop system into a continuous-time state-delay LPV system. Then, to ensure the asymptotic stability and $\mathscr{H}_\infty$ performance of the resulting closed-loop continuous-time state-delay system, a Lyapunov-Krasovskii (LK) functional is introduced with an appropriate structure to accommodate different types of delay including the intrinsic system delay and the delay imposed by input delay technique. The use of the proposed LK functional leads to a delay-dependent matrix inequality condition. Finally, using an example, we demonstrate that the proposed method can handle the sampled-data control problem.

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