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Delay-dependent $\mathscr{H}_\infty$ control for LPV systems with fast-varying time delays

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***Abstract*— Presented in this paper is a method for stabilization and induced $\mathscr{L}_2$-gain (or $\mathscr{H}_\infty$ norm) control of Linear Parameter-Varying (LPV) systems with fast-varying time delays. The Lyapunov-Krasovskii functional is used to derive the delay-dependent stability analysis and control synthesis conditions. In order to formulate the synthesis conditions as Linear Matrix Inequalities (LMIs), a relaxation method using *slack* variables is proposed. The introduction of *slack* variables along with the delay-dependent bounded real lemma conditions results in reduced conservatism with the proposed method. Sufficiency conditions that guarantee the existence of a parameter-dependent state feedback control law are then provided. Finally, numerical examples are presented to validate the proposed results and compare with recent work in the literature to show the improvements achieved.**

I. INTRODUCTION

Time delays often appear in many physical, biological and engineering systems, and in general their presence has a negative impact on the system stability. Stability analysis and control of time delay systems is a subject of great theoretical and practical importance and has been extensively examined in the controls literature [1], [2]. Stabilization and control results for time delay systems can be classified into two types: delay-independent results and delay-dependent results. Delay-independent stabilization is based on conditions that are independent of the size of the delay, guaranteeing stability for all non-negative values of time-delay. Delay-dependent criteria, ensure stabilization for magnitude of the delay smaller than a given bound. This knowledge of a bound on the size of the time delay allows to reduce the conservatism compared to the delay-independent approach. It is known that conservatism also exists in the delay-dependent approach, and though both the delay-independent and delay-dependent approaches to analyze and guarantee closed-loop stability and performance are reaching maturity, efforts are directed towards reducing the conservatism in these approaches.

Linear parameter-varying (LPV) systems with time delays often appear in many engineering systems, and at times the time-varying delay is a function of the scheduling parameter, resulting in LPV systems with parameter-varying delays. Stability analysis and control of LPV time delay systems has attracted a lot of attention in the last decade. One of the first efforts appeared in [6], where the authors analyzed a time delay LPV system and developed a delay-independent condition. State feedback controller synthesis conditions guaranteeing a desired induced $\mathscr{L}_2$ gain (or $\mathscr{H}_\infty$ norm) performance were also presented. The authors in [7] have developed stability tests for LPV time delay systems using both delay-independent and delay-dependent conditions. However, the delays were assumed to be constant and no controller synthesis conditions were provided. Improvements over the result of [6] are shown in [11] along with new results discussing the $\mathscr{L}_2 - \mathscr{L}_\infty$ gain control. Delay-dependent $\mathscr{H}_\infty$ control results for LPV systems with state delays first appeared in [12]. However, the rate information for the delay variation was not used which led to conservative results. The authors in [9] examined state feedback $\mathscr{H}_\infty$ control of LPV time-delay systems with a rate bounded time-varying delay. Their approach uses a model transformation introducing additional dynamics in the system. This shortcoming is overcome in [13], where an equivalent descriptor model transformation first introduced in [15], along with Park's inequality [16] for bounding cross terms is used to derive less conservative results. In this paper, we propose a method to reduce conservatism in the delay-dependent stability and $\mathscr{H}_\infty$ control synthesis for LPV systems with fast-varying time delays. The Lyapunov-Krasovskii functional chosen in this work avoids the use of any model transformation or any bounding of cross terms, which are the main sources of conservatism in a delay-dependent approach to time-delay systems. The only conservatism introduced by this method comes from the initial choice of the Lyapunov-Krasovskii functional and the use of Jensen's inequality employed to bound an integral term in the time derivative of the chosen Lyapunov-Krasovskii functional. A bounded-real lemma (BRL) condition guaranteeing a prescribed level of $\mathscr{H}_\infty$ performance is derived, which is then relaxed by introducing the so-called *slack* variables. Finally, sufficient conditions for the existence of a full state-feedback controller, which ensures asymptotic stability and a prescribed $\mathscr{H}_\infty$ performance level for the closed loop system is obtained in terms of an LMI optimization problem.

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II. PROBLEM STATEMENT AND PRELIMINARIES

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_h x(t-h(\rho)) + B_1 w(t) + B_2 u(t) \\ z(t) &= C_1 x(t) + C_{1h} x(t-h(\rho)) + D_{11} w(t) + D_{12} u(t) \\ x(\theta) &= \phi(\theta), \forall \theta \in [-h(\rho(0)) \;\; 0], \end{aligned} \tag{1}$$

The state-space representation of an LPV system with time delay appearing in the system state is given by (1), where $x(t) \in \mathbb{R}^n$ is the system state vector, $w(t) \in \mathbb{R}^{n_w}$ is the vector of exogenous disturbance with finite energy in the space $\mathscr{L}_2[0\ \infty)$, $u(t) \in \mathbb{R}^{n_u}$ is the control input vector, $z(t) \in \mathbb{R}^{n_z}$ is the vector of controlled outputs, $\phi(\cdot)$ denotes the initial system condition, and h is a differentiable scalar function representing the parameter-varying time-delay. The delay is assumed to be bounded and the function h lies in the set

$$\mathscr{H} := \{h \in \mathscr{C}(\mathbb{R},\mathbb{R}) : 0 \le h(t) \le h_{max} < \infty, \dot{h} \le \mu, \forall t \in \mathbb{R}_+\}$$

It should be noted, unlike the existing methods in the literature for LPV time delay systems, the restrictive assumption of bounding the delay derivative by unity does not appear here, and we only need $\mu < \infty$. The condition $\mu = 0$ refers to the constant delay case. The initial condition function ϕ is a given function in $\mathscr{C}([-h_{max}\ 0],\mathbb{R}^n)$. Wherever needed, the notation $x_t(\theta)$ is used to denote $x(t+\theta)$ for $\theta \in [-h_{max}\ 0]$. The state-space matrices $A(\cdot), A_h(\cdot), B_1(\cdot), B_2(\cdot), C_1(\cdot), C_{1h}(\cdot), C_2(\cdot), C_{2h}(\cdot), D_{11}(\cdot), D_{12}(\cdot), D_{21}(\cdot)$ are assumed to be known continuous functions of a time-varying parameter vector $\rho(\cdot) \in \mathscr{F}^{\nu}_{\mathscr{P}}$, where $\mathscr{F}^{\nu}_{\mathscr{P}}$ is the set of allowable parameter trajectories defined as

$$\mathscr{F}^{\nu}_{\mathscr{P}} \triangleq \{\rho \in \mathscr{C}(\mathbb{R}_+,\mathbb{R}^s) : \rho(t) \in \mathscr{P}, |\dot{\rho}_i(t)| \le \nu_i\}$$

where $\mathscr{P}$ is a compact subset of $\mathbb{R}^s$. Notice that the parametric dependence of the delay on ρ results in a given delay bound h_{max}, since ρ is restricted to lie in the given parameter set $\mathscr{P}$. Bounding the rate of variation of the parameter vector ρ allows the use of parameter dependent Lyapunov Krasovskii functionals resulting in less conservative analysis and synthesis conditions [3]. In this paper, we are interested in an $\mathscr{H}_\infty$ design as the performance specification for the closed-loop LPV time delay system. This paper takes advantage of a number of lemmas to prove some of the technical results. The two important ones are the Projection Lemma [4] and the Jensen's Lemma [5].

III. $\mathscr{H}_\infty$ PERFORMANCE ANALYSIS

Consider the unforced time delay LPV system (*i.e.,* $u \equiv 0$)

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_h x(t - h(\rho(t))) + B_1 w(t) \\ z(t) &= C_1 x(t) + C_{1h} x(t - h(\rho(t))) + D_{11} w(t). \end{aligned} \tag{2}$$

Theorem 1. *The LPV system (2) is asymptotically stable for all $h \in \mathscr{H}$ and satisfies the condition $\|z\|_2 \le \gamma \|w\|_2$, if there exist a continuously differentiable matrix function $P : \mathbb{R}^s \to \mathbb{S}^n_{++}$, constant matrices $Q, R \in \mathbb{S}^n_{++}$, and a scalar $\gamma > 0$ such that the following LMI*

$$\begin{bmatrix} M(\rho,\nu) & P(\rho)A_h + R & P(\rho)B_1 & C_1^T & h_{max}A^T R \\ * & -\left[1 - \sum_{i=1}^{s} \pm \left(\nu_i \frac{\partial h}{\partial \rho_i}\right)\right] Q - R & 0 & C_{1h}^T & h_{max}A_h^T R \\ * & * & -\gamma I & D_{11}^T & h_{max}B_1^T R \\ * & * & * & -\gamma I & 0 \\ * & * & * & * & -R \end{bmatrix} < 0 \tag{3}$$

holds for all $\rho \in \mathscr{F}^{\nu}_{\mathscr{P}}$, where $M(\rho,\nu) = A^T P(\rho) + P(\rho)A$ $+ \left[\sum_{i=1}^{s} \pm \left(\nu_i \frac{\partial P(\rho)}{\partial \rho_i}\right)\right] + Q - R$.

Proof. Consider the Lyapunov-Krasovskii functional $V(x_t,\rho) = V_1(x,\rho) + V_2(x_t,\rho) + V_3(x_t,\rho)$ with

$$\begin{aligned} V_1(x,\rho) &= x^T(t)P(\rho)x(t) \\ V_2(x_t,\rho) &= \int_{t-h}^{t} x^T(\xi)Qx(\xi)d\xi \\ V_3(x_t,\rho) &= \int_{-h_{max}}^{0}\int_{t+\theta}^{t} \dot{x}^T(\eta)h_{max}R\dot{x}(\eta)d\eta d\theta \end{aligned}$$

It is easy to show that $V(x_t,\rho)$ is positive definite. To ascertain the asymptotic stability of the system, the time derivative of $V(x_t,\rho)$ is computed along the trajectories of the system (2), where $\dot{V}_1(x,\rho), \dot{V}_2(x_t,\rho)$ is easily computed and

$$\dot{V}_3(x_t,\rho) = h_{max}^2 \dot{x}^T R\dot{x} - \int_{t-h_{max}}^{t} \dot{x}^T(\theta)h_{max}R\dot{x}(\theta)d\theta.$$

Since $h(t) \le h_{max}$ then

$$-\int_{t-h_{max}}^{t} \dot{x}^T(\theta)h_{max}R\dot{x}(\theta)d\theta \le -\int_{t-h}^{t} \dot{x}^T(\theta)h_{max}R\dot{x}(\theta)d\theta$$

Using Jensen's inequality, it is possible to bound the integral term in $\dot{V}_3(x_t,\rho)$ as follows:

$$\begin{aligned} \dot{V}_3(x_t,\rho) &\le h_{max}^2 \dot{x}^T R\dot{x} - \int_{t-h}^{t} \dot{x}^T(\theta)h_{max}R\dot{x}(\theta)d\theta \\ &\le h_{max}^2 \dot{x}^T R\dot{x} - \frac{h_{max}}{h}\left(\int_{t-h}^{t}\dot{x}(\theta)d\theta\right)^T R\left(\int_{t-h}^{t}\dot{x}(\theta)d\theta\right) \\ &= h_{max}^2 \dot{x}^T R\dot{x} - \frac{h_{max}}{h}[x(t) - x(t-h)]^T R[x(t) - x(t-h)] \end{aligned}$$

Finally bounding $-\frac{h_{max}}{h(t)}$ by -1 we get

$$\dot{V}_3(x_t,\rho) \le h_{max}^2 \dot{x}^T R\dot{x} - [x(t) - x(t-h)]^T R[x(t) - x(t-h)]$$

Gathering all the derivative terms and letting $\dot{V}(x_t,\rho) < 0$, results in the following inequality condition:

$$\dot{V}(x_t,\rho) \le \zeta^T(t)\Xi(\rho,\dot{\rho})\zeta(t) < 0$$

with

$$\Xi(\rho,\dot{\rho}) = \begin{bmatrix} \Xi_{11} & P(\rho)A_h + R & P(\rho)B_1 \\ * & \Xi_{22} & 0 \\ * & * & 0 \end{bmatrix} + h_{max}^2 \mathrm{T}^T R \mathrm{T}$$

$$\begin{aligned} \zeta(t) &= col[x(t),\ x(t-h),\ w(t)] \\ \mathrm{T} &= [A\ \ A_h\ \ B_1] \\ \Xi_{11} &= A^T P(\rho) + P(\rho)A + \frac{\partial P(\rho)}{\partial \rho}\dot{\rho} + Q - R \\ \Xi_{22} &= -\left(1 - \frac{\partial h}{\partial \rho}\dot{\rho}\right)Q - R \end{aligned}$$

To establish the prescribed $\mathscr{H}_\infty$ performance level γ we further require [6] $\dot{V}(x_t,\rho) - \gamma^2 w^T(t)w(t) + z^T(t)z(t) \le 0$. Substituting $z(t)$ into the inequality above leads to the inequality $\zeta^T(t)\Omega(\rho,\dot{\rho})\zeta(t) < 0$ with

$$\begin{bmatrix} -V_1-V_1^T & P-V_2^T+V_1A & -V_3^T+V_1A_h & V_1B_1 & 0 & V_1+h_{max}R \\ * & \Psi_{22}+A^TV_2^T+V_2A & R+A^TV_3^T+V_2A_h & V_2B_1 & C_1^T & V_2-P \\ * & * & \Xi_{22}+A_h^TV_3^T+V_3A_h & V_3B_1 & C_{1h}^T & V_3 \\ * & * & * & -\gamma I & D_{11}^T & 0 \\ * & * & * & * & -\gamma I & 0 \\ * & * & * & * & * & (-1-2h_{max})R \end{bmatrix} < 0 \qquad (4)$$

$$\Omega(\rho,\dot\rho) = \begin{bmatrix} \Omega_{11} & PA_h+R+h_{max}^2A^TRA_h+C_1^TC_{1h} & PB_1+h_{max}^2A^TRB_1+C_1^TD_{11} \\ * & \Xi_{22}+h_{max}^2A_h^TRA_h+C_{1h}^TC_{1h} & h_{max}^2A_h^TRB_1+C_{1h}^TD_{11} \\ * & * & D_{11}^TD_{11}-\gamma^2I \end{bmatrix}$$

where $\Omega_{11} = A^TP+PA+\frac{\partial P}{\partial\rho}\dot\rho+Q-R+h_{max}^2A^TRA+C_1^TC_1$. Applying Schur complement lemma to the above inequality expression leads to LMI (3). Finally noting that $\dot\rho$ enters affinely in the LMI, it suffices to check the LMI only at the vertices of $\dot\rho$ and hence $\frac{\partial h}{\partial\rho}\dot\rho$ and $\frac{\partial P}{\partial\rho}\dot\rho$ are replaced by $\sum_{i=1}^{s}\pm\left(\nu_i\frac{\partial h}{\partial\rho_i}\right)$ and $\sum_{i=1}^{s}\pm\left(\nu_i\frac{\partial P(\rho)}{\partial\rho_i}\right)$, respectively. The notation $\sum_{i=1}^{s}\pm(\cdot)$ is used to indicate that every combination of $+(\cdot)$ and $-(\cdot)$ should be included in the inequality. That is the inequality actually represents 2^s different inequalities that correspond to the 2^s different combinations in the summation.

A. *LMI relaxation using slack variables*

A drawback of the standard matrix inequality characterization given by Theorem 1 is that it involves multiple product terms including PA and RA and was found not to be suitable to derive the synthesis conditions. In this section, a reciprocal variant of the Projection Lemma is used to determine a relaxed condition suitable to derive the controller synthesis conditions. This technique introduces the so-called *slack* variables which bring additional flexibility in the synthesis problem. Moreover, this flexibility is expected to result in far less conservative conditions than with customary approaches.

Theorem 2. *The LPV system (2) is asymptotically stable for all $h\in\mathscr{H}$ and satisfies the condition $\|z\|_2\le\gamma\|w\|_2$ if there exist a continuously differentiable matrix function $P:\mathbb{R}^s\to\mathbb{S}^n_{++}$, constant matrices $Q,R\in\mathbb{S}^n_{++}$, matrix functions $V_1,V_2,V_3:\mathbb{R}^s\to\mathbb{R}^{n\times n}$ and a scalar $\gamma>0$ such that the LMI condition (4) at the top of this page, holds true for all $\rho\in\mathscr{F}^\nu_\mathscr{P}$, with $\Psi_{22}=\frac{\partial P}{\partial\rho}\dot\rho+Q-R$ and Ξ_{22} as defined earlier.*

Proof. The proof is inspired from [8]. We first rewrite (4) as

$$\Psi+\mathscr{C}^T\Theta^T\mathscr{D}+\mathscr{D}^T\Theta\mathscr{C}<0 \qquad (5)$$

with

$$\Psi = \begin{bmatrix} 0 & P(\rho) & 0 & 0 & 0 & h_{max}R \\ * & \Psi_{22} & R & 0 & C_1^T & -P(\rho) \\ * & * & \Xi_{22} & 0 & C_{1h}^T & 0 \\ * & * & * & -\gamma I & D_{11}^T & 0 \\ * & * & * & * & -\gamma I & 0 \\ * & * & * & * & * & (-1-2h_{max})R \end{bmatrix}$$

$$\mathscr{C} = \begin{bmatrix} -I & A & A_h & B_1 & 0 & I \end{bmatrix}$$

$$\mathscr{D} = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \end{bmatrix}$$

$$\Theta^T = \begin{bmatrix} V_1^T & V_2^T & V_3^T \end{bmatrix}$$

The explicit bases of the null-space of $\mathscr{C}$ and $\mathscr{D}$ are

$$Ker(\mathscr{C}) = \begin{bmatrix} A & A_h & B_1 & 0 & I \\ I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix} \quad Ker(\mathscr{D}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}.$$

Applying the Projection Lemma with respect to the variable Θ in (5) yields two inequalities, one of which is exactly the characterization given by (3) and the other is the following LMI

$$\begin{bmatrix} -\gamma I & D_{11}^T(\rho) & 0 \\ * & -\gamma I & 0 \\ * & * & (-1-2h_{max})R \end{bmatrix} < 0 \qquad (6)$$

The above inequality is a relaxed form of the right bottom 3×3 block of the inequality characterized by Theorem 1 and is always satisfied. Hence, the feasibility of (4) implies the feasibility of (3), which along with the result of Theorem 1 concludes the proof. The matrix functions V_1,V_2,V_3 are the so-called *slack* variables.

IV. STATE FEEDBACK $\mathscr{H}_\infty$ CONTROL SYNTHESIS

In this section, the analysis results developed in the previous section are used for the synthesis of state-feedback parameter-varying $\mathscr{H}_\infty$ controller for LPV systems with delays. For the system (1), we seek to design a parameter-dependent state-feedback controller of the form

$$u(t) = K(\rho(t))x(t) \qquad (7)$$

such that the closed loop system is asymptotically stable and has induced $\mathscr{L}_2$ norm less than γ. Using the state-feedback control law (7) results in the closed-loop system given by

$$\begin{aligned} \dot x(t) &= A_{cl}x(t)+A_hx(t-h)+B_1w(t) \\ z(t) &= C_{1,cl}x(t)+C_{1h}x(t-h)+D_{11}w(t) \end{aligned} \qquad (8)$$

where $A_{cl} = A + B_2K$ and $C_{1,cl} = C_1 + D_{12}K$. The following theorem provides a sufficiency condition for the existence of such a control law.

Theorem 3. *Given the LPV system (1), there exists a state-feedback controller of the form (7) such that the closed-loop system is asymptotically stable and satisfies the condition $\|z\|_2 \le \gamma \|w\|_2$ for all $h \in \mathscr{H}$, if there exist a continuously differentiable matrix function $\tilde{P} : \mathbb{R}^s \to \mathbb{S}^n_{++}$, constant matrices $\tilde{Q}, \tilde{R}$ and $U \in \mathbb{S}^n_{++}$, two given scalars λ_2 and $\lambda_3 \in \mathbb{R}$, a matrix function $Y : \mathbb{R}^s \to \mathbb{R}^{n_u \times n}$ and a scalar $\gamma > 0$ such that the LMI condition*

$$\begin{bmatrix} -2U & \tilde{P} - \lambda_2 U + AU + B_2 Y & -\lambda_3 U + A_h U & B_1 & 0 & U + h_{max}\tilde{R} \\ * & \widetilde{\Psi}_{22} + E & (2,3) & \lambda_2 B_1 & UC_1^T + Y^T D_{12}^T & \lambda_2 U - \tilde{P} \\ * & * & \widetilde{\Xi}_{22} + \lambda_3 (A_h U + U A_h^T) & \lambda_3 B_1 & UC_{1h}^T & \lambda_3 U \\ * & * & * & -\gamma I & D_{11}^T & 0 \\ * & * & * & * & -\gamma I & 0 \\ * & * & * & * & * & (-1 - 2h_{max})\tilde{R} \end{bmatrix} < 0 \quad (9)$$

with $(2,3) = \tilde{R} + \lambda_2 A_h U + \lambda_3 (UA^T + Y^T B_2^T), E = \lambda_2 (AU + UA^T + B_2 Y + Y^T B_2^T)$, and $\widetilde{\Psi}_{22}$, $\widetilde{\Xi}_{22}$ as defined below holds true for all $\rho \in \mathscr{F}^{\nu}_{\mathscr{P}}$. Moreover, the state-feedback control law providing a guaranteed $\mathscr{H}_\infty$ norm performance level γ is given as $u(t) = Y(\rho) U^{-1} x(t)$.

Proof. The proof is an application of Theorem 2 to the closed-loop system (8). To derive the synthesis conditions we choose the three distinct *slack* variable matrices of the analysis condition in (4) as $V_1 = V \in \mathbb{S}^n_{++}$, $V_2 = \lambda_2 V$ and $V_3 = \lambda_3 V$ where λ_2 and λ_3 are any given real scalars. Defining new variables $U = V^{-1}$ and $Y = KU$, and applying the congruence transformation using matrix $diag(U,U,U,I,I,U)$ to LMI (4) we obtain the result of Theorem 3 with $\tilde{P} = U^T P U$, $\tilde{Q} = U^T Q U$, $\tilde{R} = U^T R U$, $\widetilde{\Psi}_{22} = U^T \Psi_{22} U$ and $\widetilde{\Xi}_{22} = U^T \Xi_{22} U$. It is be noted that the inequality (9) is not an LMI unless the scalars λ_2 and λ_3 are fixed. It is observed that the results obtained are very sensitive to the scalars λ_2 and λ_3. To achieve improved results one should perform a 2-dimensional search.

V. NUMERICAL EXAMPLES

Example 1. Consider the following linear time-varying state-delayed system adopted from [9]

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & 1 + 0.2\sin(t) \\ -2 & -3 + 0.1\sin(t) \end{bmatrix} x(t) \\ &+ \begin{bmatrix} 0.2\sin(t) & 0.1 \\ -0.2 + 0.1\sin(t) & -0.3 \end{bmatrix} x(t - h(t)) \\ &+ \begin{bmatrix} 0.2 \\ 0.2 \end{bmatrix} w(t) + \begin{bmatrix} 0.2\sin(t) \\ 0.1 + 0.1\sin(t) \end{bmatrix} u(t) \\ z(t) &= \begin{bmatrix} 0 & 10 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} u(t) \end{aligned}$$

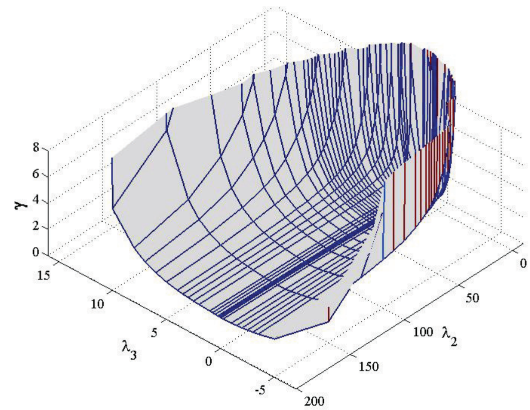

Fig. 1. γ varying with λ_2 and λ_3

Defining $\rho(t) = \sin(t)$ leads to an LPV system with the parameter space $\rho(t) \in [-1 \;\; 1]$. The control objective is to minimize the effect of disturbance $w(t)$ on the state $x_2(t)$ and maintain a reasonable control effort. The matrix D_{12} is used to penalize the control effort. To apply the synthesis results presented in Theorem 3 the parameter space is gridded uniformly using 20 points. To allow a fair comparison we choose the same parameter dependence for the controller as chosen in the control design methods [9], [13] as follows $Y(\rho) = Y_0 + \rho Y_1 + \frac{\rho^2}{2} Y_2$. The induced $\mathscr{L}_2$ performance levels with respect to different maximum time delays h_{max} and different delay variation rates μ are computed using the results of Theorem 3. As mentioned earlier, we perform a search over the two scalar variables λ_2 and λ_3. Figure 1 shows a zoomed-in plot of the obtained γ variation with changing λ_2 and λ_3. The two scalars λ_2 and λ_3 are chosen to be 1 and 7, respectively, leading to the minimum value of γ. Tables I and II present a comparison of the $\mathscr{H}_\infty$ norms obtained, whereas Table III compares the allowable maximum time-delay values using the results of this paper with those in [13] and [14]. It is evident from the tables I and II that the controller design method proposed in this paper outperforms the previous results in terms of the achievable $\mathscr{H}_\infty$ costs for same values of h_{max} and μ. In addition, it is observed that the proposed controller design methodology works for delay variation rates $\mu \geqslant 1$, whereas the past results fail. Table III shows that the maximum time-delay allowing the controller synthesis as obtained in this paper is much larger than the results in both [13] and [14]. This indicates that the method in this paper would be less conservative and allow for a larger delay range.

Example 2. This example is motivated by the control of chattering during the milling process [7], where the workpiece is clamped and fed to a rotating multi-tooth cutter as shown in Figure 2. The equations of motion for the system

TABLE I

THE RESULTING $\mathscr{H}_\infty$ NORMS FOR h_{max} = 1 (NUMBERS AS REPORTED IN THE CORRESPONDING PAPERS)

Method	$\mu=0$	$\mu=0.5$	$\mu=0.7$	$\mu=1$
[9]	6.489	6.499	6.515	Infeasible
[13]	2.129	2.239	2.531	Infeasible
Theorem 3, of this paper	1.803	1.82	1.827	1.834

TABLE II

THE RESULTING $\mathscr{H}_\infty$ NORMS FOR h_{max} = 1.5

Method	$\mu=0$	$\mu=0.5$	$\mu=0.7$	$\mu=1$
[9]	27.531	28.079	28.83	Infeasible
[13]	2.172	2.573	3.367	Infeasible
Theorem 3, of this paper	1.864	1.889	1.91	1.958

are as follows:

$$\begin{aligned} &m_1\ddot{x}_1 + k_1(x_1 - x_2) = f \\ &m_2\ddot{x}_2 + c\dot{x}_2 + k_1(x_2 - x_1) + k_2x_2 = u \\ &f = k\sin(\phi+\beta)l(t) - w \\ &l(t) = \sin(\phi)[x_1(t-h(t)) - x_1(t)] \end{aligned}$$

where k_1 and k_2 are the stiffness of the two springs, c is the damping coefficient, m_1 and m_2 are the masses of the blade and the tool, and x_1 and x_2 are the displacements of the blade and the tool, respectively. The angle β depends on the particular material and the tool used. The angle ϕ denotes the angular position of the blade, k denotes the cutting force coefficient and w denotes the disturbance. The time delay which is the time interval between two successive cuts is denoted by $h(t)$ and is approximated to be $\frac{\pi}{\omega}$ where ω is the rotation speed of the blade. The plant we are considering can be rewritten as

$$\begin{aligned} \ddot{x}_1 = &\frac{1}{m_1}[-k_1x_1 + k_1x_2 - k\sin(\phi+\beta)\sin(\phi)x_1 \\ &+ k\sin(\phi+\beta)\sin(\phi)x_1(t-h(t)) - w] \\ \ddot{x}_2 = &\frac{1}{m_2}\left[k_1x_1 - k_1x_2 - k_2x_2 - c\dot{x}_2 + u\right]. \end{aligned}$$

We consider the following problem data: $m_1 = 1$, $m_2 = 2$, $k_1 = 10$, $k_2 = 20$, $k = 2$, $c = 0.5$, $\beta = 70°$. It is noted that $\sin(\phi+\beta)\sin(\phi) = 0.1710 - 0.5\cos(2\phi+\beta)$. The system equations can be put in an LPV form with the scheduling parameter vector $\rho(t) = [\rho_1(t) \ \ \rho_2(t)]^T$, where $\rho_1(t) = \cos(2\phi+\beta)$ and $\rho_2(t) = \omega$ are measurable in real-time and can be used to develop a gain-scheduled controller. The rotation speed of the blade is assumed to be between 200 rpm and 2000 rpm, and the maximum variation rate is 1000 rpm/sec. Hence, we have $\rho_1(t) \in [-1 \ \ 1]$ and $|\frac{d\rho_1}{dt}| = |-2sin(2\phi+\beta)\omega| \leq 2\times 2000\times 2\pi/60 = 418.9(rad/sec)$, $\rho_2(t) \in [20.94 \ \ 209.4](rad/sec)$ and $|\frac{d\rho_2}{dt}| = 1000\times 2\pi/60 = 52.35(rad/sec^2)$. The delay rate $|\frac{dh(t)}{dt}| = |\frac{-\pi}{\omega^2}\times\frac{d\omega}{dt}| \leq \frac{\pi}{(200\times 2\pi/60)^2}\times 1000\times 2\pi/60 = 0.75$. We seek to design an LPV controller to attenuate the effect of the disturbance force w. The controlled variable vector z is composed of

TABLE III

THE MAXIMUM ALLOWABLE TIME-DELAY

Method	$\mu=0$	$\mu=0.5$	$\mu=0.7$
[14]	3.2	1.8	1.1
[13]	9.1	3.1	2.0
Theorem 3, of this paper	48.4	45.2	42.3

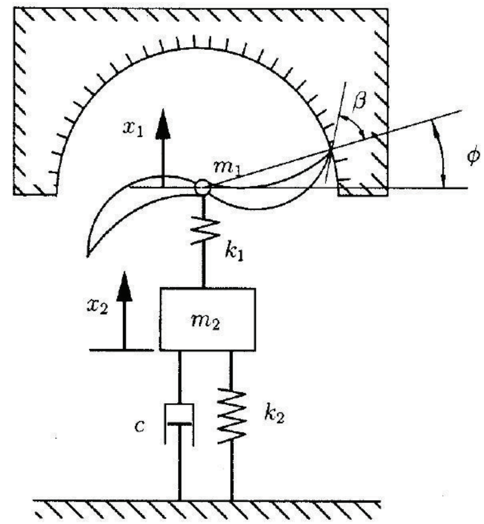

Fig. 2. Milling Process

the displacements of the two masses and the control force. Considering the state vector as $x = [x_1 \ \ x_2 \ \ \dot{x}_1 \ \ \dot{x}_2]^T$, the state-space matrices corresponding to the time-delay LPV plant to be controlled are as follows:

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -10.34+\rho_1 & 10 & 0 & 0 \\ 5 & -15 & 0 & -.25 \end{bmatrix} \quad B_1 = \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

$$A_h = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.34-\rho_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.5 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad D_{11} = \begin{bmatrix} 0 \\ 0 \\ 0.1 \end{bmatrix}$$

Note that the penalty on control effort is 0.1. We use the synthesis results presented in this paper and compare with the results obtained using the method in [9]. For simplicity both the Lyapunov matrix $\widetilde{P}$ and the *slack* variable matrix U are assumed to be constant matrices. We grid the parameter space using 5 grid points. Solving the LMI problem in Theorem 3 we obtain an $\mathscr{H}_\infty$ performance bound $\gamma = 1.031$ and using the result in [9] we have $\gamma = 1.057$. Simulations performed validate the disturbance attenuation performance of the designed controller. The disturbance $w(t)$ used in the simulation is a rectangular signal of unity magnitude for

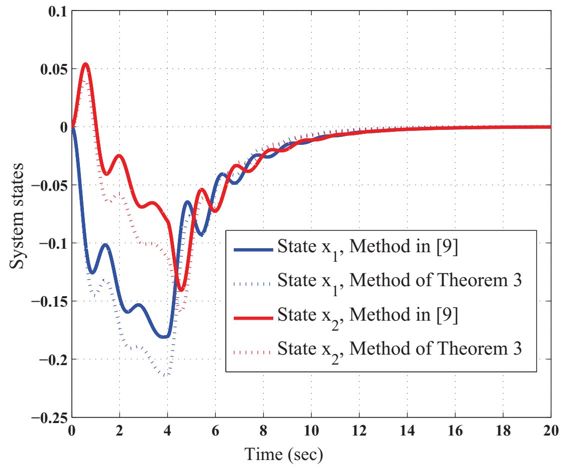

Fig. 3. Displacement of mass 1 and mass 2

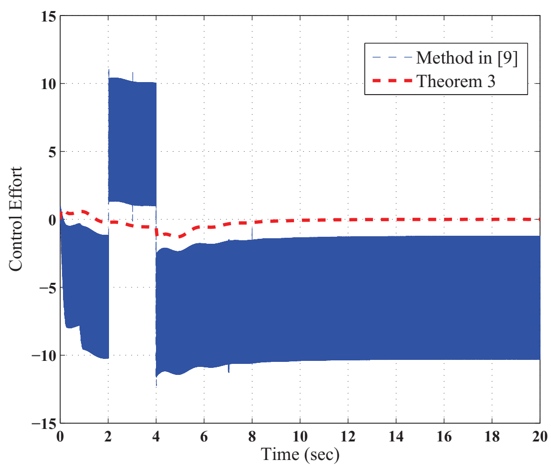

Fig. 4. Control effort for the milling process example

$0 \leq t \leq 4$ and zero elsewhere. The blade rotating speed ω starts at 200 rpm, ramps up to 1000 rpm at time t = 2sec with a slope of 800 rpm/s. The speed is held constant at 1000 rpm from time t = 3sec to 7sec, and then ramps back down to 200 rpm.Under the proposed control scheme, the control signal is shown in Figure 4 and the displacements of the two masses is shown in Figure 3. As is evident from the figures the disturbance attenuation performance using the two compared methods is similar. However, the control effort required using the method of [9] exhibits significant chatter, whereas the results of the present paper yield a much smoother control effort.

VI. CONCLUSION AND FUTURE WORK

In this paper less conservative results than existing methods in the literature for control of LPV time delay systems are developed. The presented method employs a simple Lyapunov-Krasovskii functional to derive delay-dependent analysis conditions for LPV systems with fast-varying delays. The derived analysis conditions are relaxed using a *slack* variable approach which leads to reduced conservatism in the synthesis conditions. A sufficient condition for the existence of a full state feedback controller using delay-dependent LMI conditions is proposed. Simulation results are presented to demonstrate the effectiveness of the proposed design method. It has been shown that the method presented here leads to less conservative results and can provide a controller for larger delay range. This work can further be extended to design output feedback controllers for LPV systems with time-delay and this investigation is currently ongoing.

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