2012 American Control Conference
Fairmont Queen Elizabeth, Montréal, Canada
June 27-June 29, 2012

Sampled-data Filter Design for Linear Parameter Varying Systems

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***Abstract*— In the present paper, we address the sampled-data filtering synthesis problem for continuous-time linear parameter varying (LPV) systems. The filtering error system obtained from augmenting the original LPV system and the sampled-data filter is a hybrid system. The design objective is to guarantee the system stability and a prescribed level of the induced energy-to-energy gain (or $\mathcal{H}_\infty$ norm) from the disturbance input to the estimation error. To this end, we employ the concept of *lifting* to derive an equivalent discrete-time LPV system. The discretizing method using the lifting approach is such that it would preserve the $\mathcal{H}_\infty$ norm. The viability of the proposed design method to cope with variable sampling rates will be shown through numerical examples, where reliable estimation of the LPV system outputs can be achieved.**

I. INTRODUCTION

In the past few decades, advances in the computing devices has led to the efficient ways to digitally implement the controllers and filters for the physical systems, which function in continuous-time domain [2]. In a typical hybrid process the measurable output signals are periodically sampled with an analog to digital (A/D) converter. Then, the digitized inputs are processed using a digital device (controller or filter) and will be used either in a computer program or in real world by converting to analog values. Digital implementation of the filters results in the mixture of continuous-time & discrete-time systems and signals. Due to this hybrid nature, there is a need to adapt the continuous-time filtering theory to capture this level of complexity. The issue of digital implementation was studied before primarily within the area of 'digital control theory'. In contrast to the modern sampled-data control theory, these approaches only approximately cope with the behavior of the continuous-time signals in the control system since the behavior of such systems can be captured and studied only at the sampling instants [5], [10]. Modern sampled-data control theory, on the other hand, provides an exact solution method for the analysis and synthesis of sampled-data control systems with intersample behavior taken completely into account [3]. Presented in [1] is a framework to design an $\mathcal{H}_\infty$ controller for the sampled-data systems. Using the *lifting* technique, they solved the sampled-data problem in terms of an 'equivalent' discrete-time system, where the plant is augmented by the sampler and hold devices and is lifted to a system with a finite dimensional state space representation and with infinite dimensional input and output spaces. The lifting technique was shown to preserve the input-output energy-to-energy gain with respect to the augmented plant. Authors in [8], [9] used the idea of lifting technique and applied it to the LPV systems, where they solved energy-to-energy and energy-to-peak gain problems to design state feedback and output feedback controllers. There have also been additional efforts on sampled data control design for LPV systems, *e.g.,* [4], [6]. The problem of sampled data filtering for continuous-time linear time-invariant systems has attracted considerable interest [7] and is more challenging for LPV systems.

In the present paper, we employ the idea of lifting as proposed in [1] to synthesize a discrete-time filter for an LPV continues-time system. Similar to the controller design procedure in [9], we lift the continuous-time system and find its equivalent discrete-time representation. Then, we design a discrete-time filter to solve the energy-to-energy gain problem for the filtering error system.

The notation used in this paper is standard. $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+$ is the set of non-negative real numbers. $\mathbb{R}^n$ and $\mathbb{R}^{n\times n}$ are used to denote the set of real vectors of dimension n and the set of real $n \times n$ matrices, respectively. Given a symmetric matrix $X = X^T \in \mathbb{R}^{n\times n}$, $X > 0$ $(X \geq 0)$ denotes matrix positive definiteness (semi-definiteness). The notation $(\cdot)^T$ denotes the transpose of a real matrix. Given a real $n \times m$ matrix Y with rank r, the orthogonal complement $Y^\perp$ is defined as the $(n-r)\times n$ matrix that satisfies $Y^\perp Y = 0$ and $Y^\perp Y^{\perp\top} > 0$. The $\mathcal{L}_2[a,b]$-norm of a continuous-time signal is defined as $\|f\|_{\mathcal{L}_2[a,b]} = (\int_b^a |f(t)|^2)^{\frac{1}{2}}$. The space of the time series with finite $\mathcal{L}_2[a,b]$ norm is called signal space $\mathcal{L}_2[a,b]$. The l_2-norm of a discrete signal is defined as $\|f\|_{l_2} = (\sum_{k=0}^{\infty} |f(k)|^2)^{\frac{1}{2}}$. Finally, $(\cdot)^*$ denotes the adjoint of an operator on the Hilbert space.

II. PRELIMINARIES AND PROBLEM STATEMENT

Consider a stable n^{th}-order LPV system with the following state space realization

$$\begin{aligned} \dot{x}(t) &= A(\rho(t))x(t) + B(\rho(t))w(t) \\ y(t) &= C(\rho(t))x(t) + D(\rho(t))w(t) \\ z(t) &= L(\rho(t))x(t) \end{aligned} \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $y(t) \in \mathbb{R}^{n_y}$ is the measured output, $z(t) \in \mathbb{R}^{n_z}$ is the signal to be estimated, and $w(t) \in \mathbb{R}^{n_w}$ is disturbance vector containing both process and measurement noise. The system matrices $A(\cdot), B(\cdot), C(\cdot), D(\cdot)$ and $L(\cdot)$ are real continuous functions of a time varying parameter vector $\rho(t)$ and of appropriate dimensions. The parameter vector $\rho(t) \in \mathcal{F}_{\mathcal{P}}^{v}$ is assumed to

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978-1-4577-1096-4/12/$26.00 ©2012 AACC

be measurable in real-time, where $\mathcal{F}_{\mathcal{P}}^{v}$ is the set of allowable parameter trajectories defined as

$$\mathcal{F}_{\mathcal{P}}^{v} \triangleq \{\rho : \rho \in C(\mathbb{R}, \mathbb{R}^s) : \rho(t) \in \mathcal{P}, |\dot{\rho}_i(t)| \leq v_i, \\ i = 1, 2, ..., s \in \mathbb{R}_+\}$$

where $\mathcal{P}$ is a compact subset of $\mathbb{R}^s$, and $\{v_i\}_{i=1}^{s}$ are nonnegative numbers. In the present paper, the objective is to design a stable discrete-time filter that uses the discrete samples of $y(t)$ as input and provides an estimate of $z(t)$. The time intervals are considered as $[0, t_1)$, $[t_1, t_2)$, $\ldots$, $[t_k, t_{k+1})$, $\ldots$ that are not necessarily equi-spaced with t_k's being the sampling instants. For the sake of brevity, throughout the paper, k will be used to represent t_k, and the length of the k^{th} interval will be represented by τ_k, that is, $\tau_k = t_{k+1} - t_k$. To this aim consider an n^{th}-order discrete-time parameter-varying filter $\mathcal{F}$ represented by the following state space description

$$\begin{aligned} \xi_d(k+1) &= G_d(\rho(k))\xi_d(k) + H_d(\rho(k))y_d(k) \\ \hat{z}_d(k) &= J_d(\rho(k))\xi_d(k) + K_d(\rho(k))y_d(k) \end{aligned} \quad (2)$$

with the system matrices of appropriately defined dimensions. In (2), ξ_d, y_d and $\hat{z}_d$ represent the discrete-time filter state vector, the discrete samples of measurement data, *i.e.,* $y_d(k) = y(t_k)$, and the system output estimates, respectively. By means of a zero order hold as D/A converter, the continuous piecewise signal $\hat{z}(t)$ is obtained as $\bar{z}(t) = z_d(t_k)$ for $t_k \leq t < t_{k+1}$. Figure 1 shows the configuration of the system under study, the interconnections of the open loop continuous-time system and the discrete-time filter, along with the signal conversion devices. We assume in that the A/D converter is an ideal sampler and that the quantization errors are neglected.

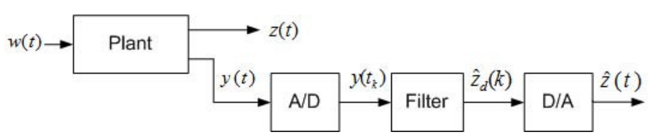

Fig. 1. The block diagram of hybrid system

Defining the filtering error as $e(t) = z(t) - \hat{z}(t)$, the objective is to design a sub-optimal discrete-time filter $\mathcal{F}$ represented by (2) such that

- The *filtering error system* resulted from augmenting (1) and (2) is asymptotically stable, and
- The induced $\mathcal{L}_2$-gain (or $\mathcal{H}_\infty$ norm) of the filtering error system from the disturbance w to the estimation error e is less than a prescribed performance level γ.

The filter design problem described above is a *hybrid filtering problem*, where the physical system has a continuous dynamics, while the filter to estimate the plant outputs is implemented in digital computer. It is noted that while the open loop continuous-time system matrices depend on the continuously varying LPV parameter vector $\rho(t)$, the discrete-time filter matrices vary with respect to the bounded piecewise constant vector $\rho(k)$ that is determined from $\rho(t)$ using the procedure described below. The parameter vector $\rho(k) \in \mathcal{E}_{\mathcal{P}}^{v}$ is measurable at t_k for $k = 0, 1, 2, \ldots$ and $\mathcal{E}_{\mathcal{P}}^{v}$ is the set of allowable parameter trajectories defined as

$$\begin{aligned} \mathcal{E}_p^v \triangleq \{\rho : \rho(t) &\in \mathcal{P},\ \rho(t_k + t) = \rho(t_k), \\ t &\in [0, \tau_k),\ \ |\rho_i(t_{k+1}) - \rho_i(t_k)| \leq v_i, \\ k &\in \mathbb{Z}^+,\ \ i = 1, 2, ..., s \in \mathbb{R}_+\} \end{aligned}$$

Before proposing the solution to the afore-discussed problem, in this part we state the problem of designing parameter-varying filters for the discrete-time LPV systems and its solution. The LMI-based solution to this problem is extension of those in [6] to LPV systems. Consider a discrete LPV system represented by

$$\begin{aligned} x_d(k+1) &= A_d(\rho(k))x_d(k) + B_d(\rho(k))w_d(k) \\ y_d(k) &= C_d(\rho(k))x_d(k) \\ z_d(k) &= L_d(\rho(k))x_d(k) \end{aligned} \quad (3)$$

with the objective of designing a suboptimal discrete-time filter $\mathcal{F}$ represented by (2) to make the $\mathcal{H}_\infty$ norm of the error system from the disturbance w_d to the estimation error e_d defined by $e_d(k) = z_d(k) - \hat{z}_d(k)$, less than a prescribed performance level, that is

$$\sup_{\rho \in \mathcal{E}_p^v} \sup_{w_d \in l_2 - \{0\}} \frac{\|e_d\|_{l_2}}{\|w_d\|_{l_2}} < \gamma$$

where γ is a given positive scalar. The following Theorem serves to solve the aforementioned γ-suboptimal problem.

Theorem 1: There exists an n^{th}-order filter $\mathcal{F}$ to solve the γ-suboptimal filtering problem if and only if there exist matrices $X(\rho(k)) > 0$ and $Y(\rho(k)) > 0$ such that there is a feasible solution to the following set of LMIs

$$\begin{bmatrix} C_d^T \\ 0 \end{bmatrix}^{\perp} \left(\begin{bmatrix} Y(\rho(k-1)) & 0 \\ 0 & \gamma^2 I \end{bmatrix} - \begin{bmatrix} A_d & B_d \\ L_d & 0 \end{bmatrix}^T \begin{bmatrix} Y(\rho(k)) & 0 \\ 0 & \gamma^2 I \end{bmatrix} \begin{bmatrix} A_d & B_d \\ L_d & 0 \end{bmatrix} \right) \begin{bmatrix} C_d^T \\ 0 \end{bmatrix}^{\perp T} > 0 \quad (4)$$

$$\begin{bmatrix} X(\rho(k-1)) & 0 \\ 0 & \gamma^2 I \end{bmatrix} - \begin{bmatrix} A_d^T \\ B_d^T \end{bmatrix} X(\rho(k)) \begin{bmatrix} A_d & B_d \end{bmatrix} > 0 \quad (5)$$

$$Y(\rho(k)) \geq X(\rho(k)). \quad (6)$$

Once X and Y are found, Y_{11}, Y_{12} and Y_{22} are determined, we form the factorization problem $Y - X = Y_{12}Y_{22}^{-1}Y_{12}^T$ using SVD and define

$$P(\rho(k)) = \begin{bmatrix} Y(\rho(k)) & Y_{12}(\rho(k)) \\ Y_{12}^T(\rho(k)) & Y_{22}(\rho(k)) \end{bmatrix}.$$

Finally, the filter matrices in F can be computed by solving the basic LMI

$$\left(\begin{bmatrix} \tilde{A} & \tilde{D} \\ \tilde{C} & 0 \end{bmatrix} + \begin{bmatrix} \tilde{B} \\ \tilde{H} \end{bmatrix} F \begin{bmatrix} \tilde{M} & 0 \end{bmatrix} \right)^T \begin{bmatrix} P(\rho(k)) & 0 \\ 0 & I \end{bmatrix} \left(\begin{bmatrix} \tilde{A} & \tilde{D} \\ \tilde{C} & 0 \end{bmatrix} + \begin{bmatrix} \tilde{B} \\ \tilde{H} \end{bmatrix} F \begin{bmatrix} \tilde{M} & 0 \end{bmatrix} \right) < \begin{bmatrix} P(\rho(k-1)) & 0 \\ 0 & \gamma^2 I \end{bmatrix} \quad (7)$$

for F with the system matrices given by

$$\begin{aligned}\tilde{A} &= \begin{bmatrix} A_d(\rho(k)) & 0 \\ 0 & 0 \end{bmatrix}, & \tilde{B} &= \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \\ \tilde{M} &= \begin{bmatrix} C_d(\rho(k)) & 0 \\ 0 & I \end{bmatrix}, & \tilde{D} &= \begin{bmatrix} B_d(\rho(k)) \\ 0 \end{bmatrix} \\ \tilde{C} &= \begin{bmatrix} L_d(\rho(k)) & 0 \end{bmatrix}, & \tilde{H} &= \begin{bmatrix} -I & 0 \end{bmatrix} \\ \tilde{F} &= \begin{bmatrix} K_d(\rho(k)) & J_d(\rho(k)) \\ H_d(\rho(k)) & G_d(\rho(k)) \end{bmatrix}. \end{aligned}$$

III. Sampled-data filter design using lifting method

In this section, we first describe the details about the idea behind lifting method that we use in this paper to discretize the continuous-time system and use it further for the discrete-time LPV filter design problem.

We consider a signal $f(t) \in \mathcal{L}_2[0,\infty)$. By "signal lifting", we mean breaking up $f(t)$ into the intervals $[0,t_1),[t_1,t_2),\ldots,[t_k,t_{k+1}),\ldots$ and making a sequence of signals denoted by $f_k(t)$, whose elements are defined as $f_k(t) = f(t_k + t)$ for $0 \le t < t_{k+1} - t_k$ or equivalently $0 \le t < \tau_k$. It is evident that $f_k(t) \in \mathcal{L}_2[0,\tau_k)$. In the following, we extend the definition of lifting to the LPV systems. Consider the continuous-time LPV system Δ

$$\Delta : \begin{cases} \dot{x}(t) = \mathfrak{A}(\rho(t))x(t) + \mathfrak{B}(\rho(t))w(t) \\ q(t) = \mathfrak{C}(\rho(t))x(t) + \mathfrak{D}(\rho(t))w(t). \end{cases} \tag{8}$$

One can think of Δ as an operator acting on the input signal w to give the output signal q, *i.e.,* $q(t) = (\Delta w)(t) = \int_0^t \Delta(t,s)w(s)ds$, where $\Delta(t,s)$ is the kernel function associated with the non-autonomous system Δ. The lifting of the system Δ is the process of finding an operator denoted by $\hat{\Delta}$ that maps the lifted input signal $w_k(t)$ to the lifted output signal $q_k(t)$ in the sense that both the original and lifted systems have equivalent $\mathcal{H}_\infty$ norm. In the following, we derive the state space realization describing the lifted system $\hat{\Delta}$. The integral solution to the state space representation (8) yields

$$\begin{aligned} x(t_k+t) &= \Phi(t_k+t,t_k)x(t_k) \\ &+ \int_{t_k}^{t_k+t} \Phi(t_k+t,s)\mathfrak{B}(\rho(s))w(s)ds \end{aligned} \tag{9}$$

for $t \in [0,\tau_k)$, where

$$\Phi(t_2,t_1) = e^{\int_{t_1}^{t_2} \mathfrak{A}(\rho(\zeta))d\zeta} \tag{10}$$

is the state transition matrix. Referring to the lifted signal definition, we have $\rho_k(t) = \rho(t_k+t)$ for $t \in [0,\tau_k)$. Changing the integral variable $\zeta' = \zeta - t_k$ in the definition of state transition matrix results in

$$\begin{aligned} \Phi(t_k+t_2,t_k+t_1) &= e^{\int_{t_k+t_1}^{t_k+t_2} \mathfrak{A}(\rho(\zeta))d\zeta} \\ &= e^{\int_{t_1}^{t_2} \mathfrak{A}(\rho(t_k+\zeta'))d\zeta'} \\ &= e^{\int_{t_1}^{t_2} \mathfrak{A}(\rho_k(\zeta'))d\zeta'} \end{aligned}$$

$(0 \le t_1 \le t_2 < \tau_k)$ which we call $\Phi_k(t_2,t_1)$ hereafter. Equation (9) can be simplified if we change the integral variable and use the lifted signal definition as

$$x_k(t) = \Phi_k(t,0)x_k(0) + \int_0^t \Phi_k(t,s)\mathfrak{B}(\rho_k(s))w_k(s)ds$$

for $t \in [0,\tau_k)$. Similarly the lifted output signal is

$$\begin{aligned} q_k(t) &= \mathfrak{C}(\rho_k(t))\left\{\Phi_k(t,0)x_k(0) + \int_0^t \Phi_k(t,s)\mathfrak{B}(\rho_k(s))w_k(s)ds\right\} \\ &+ \mathfrak{D}(\rho_k(t))w_k(t). \end{aligned}$$

Finally, we can represent the state space realization of $\hat{\Delta}$ for $t \in [0,\tau_k)$ as

$$\begin{aligned} x_{k+1}(0) &= \Phi_k(\tau_k,0)x_k(0) + \int_0^{\tau_k} \Phi_k(\tau_k,s)\mathfrak{B}(\rho_k(s))w_k(s)ds \\ q_k(t) &= \mathfrak{C}(\rho_k(t))\Phi_k(t,0)x_k(0) \\ &+ \int_0^{\tau_k} \{\mathfrak{C}(\rho_k(t))\Phi_k(t,s)\mathbf{1}(t-s)\mathfrak{B}(\rho_k(s)) \\ &+ \mathfrak{D}(\rho_k(s))\delta(t-s)\}w_k(s)ds \end{aligned} \tag{11}$$

where

$$1(t) = \begin{cases} 1 & t \ge 0 \\ 0 & t < 0 \end{cases}$$

is the unit step function that enables us to slide the upper bound of the second integral in (11) from t to τ_k and the shift property of Kroneker delta helps absorb feed through matrix into the integral. The operator $\hat{\Delta}$ can now be represented by

$$\hat{\Delta} : \begin{cases} x_{k+1} = \mathfrak{A}_k x_k + \mathfrak{B}_k w_k \\ q_k = \mathfrak{C}_k x_k + \mathfrak{D}_k w_k \end{cases} \tag{12}$$

where the operator

$$\mathfrak{A}_k : \mathbb{R}^n \to \mathbb{R}^n$$

is defined by

$$\mathfrak{A}_k f = \Phi_k(\tau_k,0)f.$$

The operator

$$\mathfrak{B}_k : \mathcal{L}_2[0,\tau_k) \to \mathbb{R}^n$$

has a kernel function that defines the following integral transformation

$$\mathfrak{B}_k f = \int_0^{\tau_k} \Phi_k(\tau_k,s)\mathfrak{B}(\rho_k(s))f(s)ds. \tag{13}$$

The operator

$$\mathfrak{C}_k : \mathbb{R}^n \to \mathcal{L}_2[0,\tau_k)$$

for $t \in [0,\tau_k)$ is defined by

$$(\mathfrak{C}_k f)(t) = \mathfrak{C}(\rho_k(t))\Phi_k(t,0)f,$$

and the operator

$$\mathfrak{D}_k : \mathcal{L}_2[0,\tau_k) \to \mathcal{L}_2[0,\tau_k)$$

defines the following integral transformation for $t \in [0,\tau_k)$

$$\begin{aligned} (\mathfrak{D}_k f)(t) = \int_0^{\tau_k} &\{\mathfrak{C}(\rho_k(s))\Phi_k(t,s)1(t-s)\mathfrak{B}(\rho_k(s)) \\ &+ \mathfrak{D}(\rho_k(s))\delta(t-s)\}f(s)ds. \end{aligned} \tag{14}$$

The lifted system (12) has infinite-dimensional input and output spaces but its state space realization is finite-dimensional with the dimension of the original system. The question that arises is how we can model the lifted system (12) using a discrete-time LPV system such that the stability and an upper bound on the $\mathcal{H}_\infty$ norm is preserved. Indeed, at this stage we seek for a model of the lifted system to be implementable in digital computer by picking up the input and parameter signals in discrete time instants. To this aim, we first augment the lifted system with an operator to remove the feed through operator $\mathfrak{D}_k$. Notice that this operator maps the infinite dimensional input space to the infinite dimensional output space. We define the following unitary operator

$$\Sigma = \begin{bmatrix} -\mathfrak{D}_k & (\sigma^2 I - \mathfrak{D}_k\mathfrak{D}_k^*)^{1/2} \\ (\sigma^2 I - \mathfrak{D}_k^*\mathfrak{D}_k)^{1/2} & \mathfrak{D}_k^* \end{bmatrix} \tag{15}$$

and construct a new system $\bar{\Delta}$ by upper linear fractional transformation (LFT) augmentation of the lifted system (12) with (15). It can be shown that the lifted system and the new augmented system have the same stability property and $\mathcal{H}_\infty$ norm bound provided that $\mathfrak{D}_k^*\mathfrak{D}_k - \sigma^2 I < 0$ [9]. After some intermediate calculations, we determine the following state space representation for the augmented lifted system denoted by $\bar{\Delta}$

$$\bar{\Delta} : \begin{cases} x_{k+1} = \bar{\mathfrak{A}}_k x_k + \bar{\mathfrak{B}}_k w_k \\ \bar{q}_k = \bar{\mathfrak{C}}_k x_k \end{cases} \tag{16}$$

where

$$\begin{aligned} \bar{\mathfrak{A}}_k &= \mathfrak{A}_k + \mathfrak{B}_k\mathfrak{D}_k^*(\sigma^2 I - \mathfrak{D}_k\mathfrak{D}_k^*)^{-1}\mathfrak{C}_k \\ \bar{\mathfrak{B}}_k &= \sigma\mathfrak{B}_k(\sigma^2 I - \mathfrak{D}_k^*\mathfrak{D}_k)^{-1/2} \\ \bar{\mathfrak{C}}_k &= \sigma(\sigma^2 I - \mathfrak{D}_k\mathfrak{D}_k^*)^{-1/2}\mathfrak{C}_k \end{aligned} \tag{17}$$

or equivalently for the last two expressions

$$\begin{aligned} \bar{\mathfrak{B}}_k\bar{\mathfrak{B}}_k^* &= \sigma^2\mathfrak{B}_k(\sigma^2 I - \mathfrak{D}_k^*\mathfrak{D}_k)^{-1}\mathfrak{B}_k^* \\ \bar{\mathfrak{C}}_k^*\bar{\mathfrak{C}}_k &= \sigma^2\mathfrak{C}_k^*(\sigma^2 I - \mathfrak{D}_k\mathfrak{D}_k^*)^{-1}\mathfrak{C}_k. \end{aligned}$$

Although (16) is a finite dimensional system, but its elements are expressed in terms of operators. In the sequel, we compute the operator compositions analytically and construct a discrete LPV system with the following state space representation

$$\begin{aligned} x_d(k+1) &= \mathfrak{A}_d(\rho(k))x_d(k) + \mathfrak{B}_d(\rho(k))w_d(k) \\ q_d(k) &= \mathfrak{C}_d(\rho(k))x_d(k). \end{aligned} \tag{18}$$

The matrices in (18) are obtained as

$$\begin{aligned} \mathfrak{A}_d &= \Upsilon_{22}(\tau_k,0) - \Upsilon_{21}(\tau_k,0)\Upsilon_{11}^{-1}(\tau_k,0)\Upsilon_{12}(\tau_k,0) \\ \mathfrak{B}_d(\rho(k))\mathfrak{B}_d^*(\rho(k)) &= \sigma^2\Pi_{21}(\tau_k,0)\Pi_{11}^{-1}(\tau_k,0) \\ \mathfrak{C}_d^*(\rho(k))\mathfrak{C}_d(\rho(k)) &= -\sigma^2\Upsilon_{11}^{-1}(\tau_k,0)\Upsilon_{12}(\tau_k,0) \end{aligned} \tag{19}$$

where the matrices $\Pi_{ij}(t,0)$ and $\Upsilon_{ij}(t,0)$ for $i,j \in \{1,2\}$ and $t \in [0,\tau_k)$ are the elements of the state transition matrices defined by

$$\begin{aligned} \Pi(t,0) &= \begin{bmatrix} \Pi_{11}(t,0) & \Pi_{12}(t,0) \\ \Pi_{21}(t,0) & \Pi_{22}(t,0) \end{bmatrix} \\ &= exp\left\{\int_0^t \begin{bmatrix} -A^T & -C^TC \\ \frac{1}{\sigma^2}BB^T & A \end{bmatrix} d\zeta\right\} \end{aligned}$$

and

$$\begin{aligned} \Upsilon(t,0) &= \begin{bmatrix} \Upsilon_{11}(t,0) & \Upsilon_{12}(t,0) \\ \Upsilon_{21}(t,0) & \Upsilon_{22}(t,0) \end{bmatrix} \\ &= exp\left\{\int_0^t \begin{bmatrix} -A^T & -\frac{1}{\sigma^2}C^TC \\ BB^T & A \end{bmatrix} d\zeta\right\} \end{aligned}$$

in which A, B and C are system matrices corresponding to the continuous-time plant (1) and depend on $\rho_k(\cdot)$. It has been proven in [9] that if the discrete-time LPV system (18) has an $\mathcal{H}_\infty$ norm less than σ, the operators $\bar{\Delta}$, $\hat{\Delta}$ and Δ will also have an $\mathcal{H}_\infty$ norm less than σ. Therefore, the scalar σ indicates how close the lifted system $\bar{\Delta}$ to the original system Δ can be in terms of energy-to-energy gain of the systems. The proof of the second sub-equation of (19) can be found in appendix. The other equations can be proven similarly, and hence the proofs are not shown here due to the space limitation.

Remark 1: $\mathfrak{B}_d(\rho(k))$ and $\mathfrak{C}_d(\rho(k))$ can be determined using SVD or through Cholskey factorization of $\mathfrak{B}_d(\rho(k))\mathfrak{B}_d^*(\rho(k))$ and $\mathfrak{C}_d^*(\rho(k))\mathfrak{C}_d(\rho(k))$ calculated in (19).

Remark 2: It can be shown that $\mathfrak{D}_k^*\mathfrak{D}_k - \sigma^2 I < 0$ (which is necessary to lift a system and determine (16)) if and only if $\Pi_{11}(\tau_k,0)$ (or equivalently $\Pi_{22}(\tau_k,0)$) is invertible.

Summary of the Sampled-data Filter Design Procedure

Shown in Figure 2 is the schematic of different steps involved in the design of a discrete-time filter for a continuous-time LPV system using the lifting method. As shown in the figure, the first step is to augment the outputs of the continuous-time plant (1) and convert it to the representation in (8), that is $q(t) = col(z(t),y(t))$. Next, following the procedure described earlier in this section, we first lift the system and find an system in the form (12). Another transformation is then applied to convert the lifted system to one with zero direct feed through matrix. Since the obtained system is expressed in terms of the operators, we find a discrete-time equivalent system instead. In the corresponding block in this Figure, the discrete-time system output $q_d(k)$ includes the discretized version of both outputs $z(t)$ and $y(t)$ corresponding to the original plant (1), i.e., $q_d(k) = col(z_d(k),y_d(k))$. Using the results presented in Theorem 1, we then design a discrete-time LPV filter to estimate $z_d(k)$ using the information from $y_d(k)$. The discrete-time filter designed at the final step is the 'filter block' in Figure 1.

IV. SIMULATION RESULTS

In this section, we present numerical results obtained from applying the proposed sampled-data LPV filter design method for a continuous-time system using lifting method.

We consider the following LPV continuous-time system

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} 0 & 1+0.2sin(t) \\ -2 & -3+0.1sin(t) \end{bmatrix} x(t) + \begin{bmatrix} 0.2 \\ -0.2 \end{bmatrix} w(t) \\ z(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix} x(t) \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x(t). \end{aligned}$$

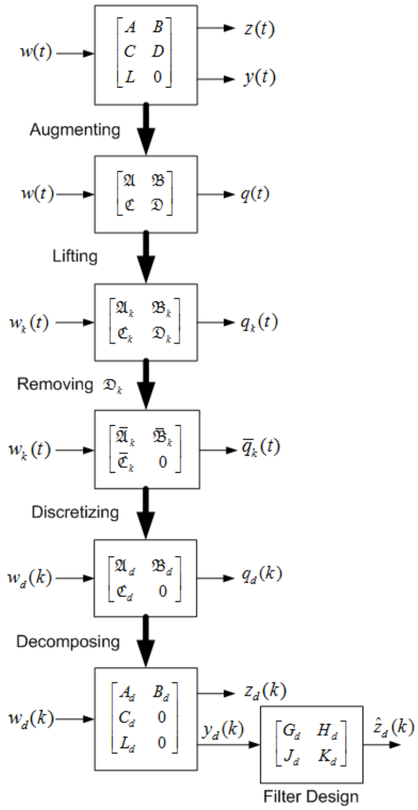

Fig. 2. Hierarchical filter design

The objective is to find an LPV sampled-data filter in order to estimate the second state variable using the measurement of the first state. We assume the sine term in the above model corresponds to a model parameter, whose functional representation is not known *a priori* but can be measured in real time. Hence, we define $\rho(t) = sin(t)$ and reformulate the original system to obtain an LPV state space representation with the parameter space $\rho \in [-1, 1]$ and $v = 1$. It is assumed that the system is affected by an input disturbance signal, with $w(t) = 1$ for $t \in [0, 5]$. We consider three designs corresponding to different sampling rates. First, we design a discrete-time filter for the case of a constant sampling rate $\tau_k = 0.1$. The red thin solid curve in Figure 3 illustrates the estimation result along with the actual output of the continuous-time system (the blue dashed curve). It is noted that the output tracking is even improved for lower sampling rates than $\tau_k = 0.1$. Next, we examine the case of a constant sampling rate $\tau_k = 0.4$. The black thick solid curve in Figure 3 shows the results confirming that even with a large sampling rate, the designed filter can provide reliable estimates of the output of the continuous-time LPV system. Finally, we consider the case of a variable sampling rate, in which the sampling rate varies corresponding to the pattern $\{0.3, 0.4, 0.5, 0.3, 0.4, 0.5, \ldots\}^{sec}$ Starting from $t = 0$, the pattern above is associated with the sampling at the time instants $\{0, 0.3, 0.7, 1.2, 1.5, 1.9, 2.4, \ldots\}^{sec}$. Figure 4 shows acceptable tracking result that the sampled-data LPV filter can provide. As simulation outcomes demonstrate, the proposed method exhibits promising results in terms of output tracking even for high constant sampling rates and variable sampling rates.

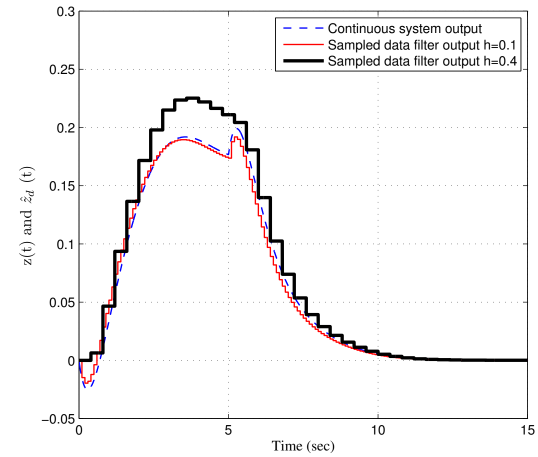

Fig. 3. The estimation results for the sampling rate of 0.1 sec and 0.4 sec

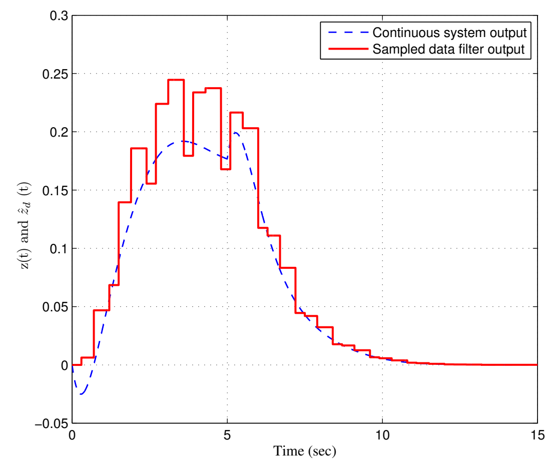

Fig. 4. The estimation results for a variable sampling rate

V. CONCLUDING REMARKS

In this paper, we presented a sampled-data filter design method for stable continuous-time LPV systems using the lifting technique. The design method consisted of few intermediate steps to achieve a discrete-time LPV system by employing the lifting method. The discretized system determined from lifting was shown to have the same $\mathcal{H}_\infty$ norm bound as the original LPV system. Finally, the discrete-time LPV filter was designed for the discretized system and was shown to effectively handle large and even variable sampling rates. In contrast to the continuous-time filtering methods for LPV systems, the design method led to solving only a finite number of LMIs due to the discrete nature of the procedure. Numerical results validate the viability of the proposed sampled-data filtering method.

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APPENDIX

The results presented here are extension of those in [1] for the LPV systems. Considering the operator $\mathfrak{D}_k$ in (14), the relation $f = \mathfrak{D}_k u$ can be described by the following state space equations:

$$\begin{aligned} \dot{x}_1(t) &= A(\rho_k(t))x_1(t) + B(\rho_k(t))u(t) \\ f(t) &= C(\rho_k(t))x_1(t) \\ x_1(0) &= 0, \qquad 0 \le t \le \tau_k \end{aligned} \tag{20}$$

and the state space representation for $y = \mathfrak{D}_k^* f$ is

$$\begin{aligned} \dot{x}_2(t) &= -A^T(\rho_k(t))x_2(t) - C^T(\rho_k(t))f(t) \\ y(t) &= B^T(\rho_k(t))x_2(t) \\ x_2(\tau_k) &= 0, \qquad 0 \le t \le \tau_k \end{aligned} \tag{21}$$

Combining (20) and (21), one can obtain the following state space representation for $y = (\sigma^2 I - \mathfrak{D}_k^*\mathfrak{D}_k)u$

$$\begin{aligned} \begin{bmatrix}\dot{x}_2(t)\\ \dot{x}_1(t)\end{bmatrix} &= \begin{bmatrix}-A^T(\rho_k(t)) & -C^T(\rho_k(t))C(\rho_k(t))\\ 0 & A(\rho_k(t))\end{bmatrix} \\ &\quad \begin{bmatrix}x_2(t)\\ x_1(t)\end{bmatrix} + \begin{bmatrix}0\\ -B(\rho_k(t))\end{bmatrix}u(t) \\ y(t) &= \begin{bmatrix}B^T(\rho_k(t)) & 0\end{bmatrix}\begin{bmatrix}x_2(t)\\ x_1(t)\end{bmatrix} + \sigma^2 u(t) \\ \begin{bmatrix}x_2(\tau_k)\\ x_1(0)\end{bmatrix} &= 0, \qquad 0 \le t \le \tau_k \end{aligned} \tag{22}$$

Notice that this is a two point boundary condition differential equation. Rewriting (22) for u in terms of y, the inverse operator composition $u = (\sigma^2 I - \mathfrak{D}_k^*\mathfrak{D}_k)^{-1}y$ is obtained as

$$\begin{aligned} \begin{bmatrix}\dot{x}_2(t)\\ \dot{x}_1(t)\end{bmatrix} &= \begin{bmatrix}-A^T(\rho_k(t)) & -C^T(\rho_k(t))C(\rho_k(t))\\ \sigma^{-2}B(\rho_k(t))B^T(\rho_k(t)) & A(\rho_k(t))\end{bmatrix} \\ &\quad \begin{bmatrix}x_2(t)\\ x_1(t)\end{bmatrix} + \begin{bmatrix}0\\ \sigma^{-2}B(\rho_k(t))\end{bmatrix}y(t) \\ u(t) &= \begin{bmatrix}\sigma^{-2}B^T(\rho_k(t)) & 0\end{bmatrix}\begin{bmatrix}x_2(t)\\ x_1(t)\end{bmatrix} + \sigma^{-2}y(t) \\ \begin{bmatrix}x_2(\tau_k)\\ x_1(0)\end{bmatrix} &= 0, \qquad 0 \le t \le \tau_k \end{aligned} \tag{23}$$

The state transition matrix corresponding to (23) is

$$\begin{aligned} \Pi(t_2,t_1) &= \begin{bmatrix}\Pi_{11}(t_2,t_1) & \Pi_{12}(t_2,t_1)\\ \Pi_{21}(t_2,t_1) & \Pi_{22}(t_2,t_1)\end{bmatrix} \\ &= exp\left\{\int_{t_1}^{t_2}\begin{bmatrix}-A^T & -C^TC\\ \sigma^{-2}BB^T & A\end{bmatrix}d\zeta\right\} \end{aligned}$$

where the system matrices are dependent on $\rho_k(\cdot)$. It can be shown the inverse exists if and only if $\Pi_{11}(\tau_k,0)$ (or equivalently $\Pi_{22}(\tau_k,0)$) is invertible. The solution of (23) for the state vector is

$$\begin{bmatrix}x_2(t)\\ x_1(t)\end{bmatrix} = \Pi(t,0)\begin{bmatrix}x_2(0)\\ x_1(0)\end{bmatrix} + \int_0^t \Pi(t,s)\begin{bmatrix}0\\ \sigma^{-2}B(\rho_k(s))\end{bmatrix}y(s)ds \tag{24}$$

If we replace $t = \tau_k$ in (24) and use the boundary conditions $x_2(\tau_k) = x_1(0) = 0$, we obtain $x_2(0)$. Solving $u(t)$ in (23), the kernel of the operator $(\sigma^2 I - \mathfrak{D}_k^*\mathfrak{D}_k)^{-1}$ is given by

$$\begin{aligned} &(\sigma^2 I - \mathfrak{D}_k^*\mathfrak{D}_k)^{-1}(t,s) = \\ &\begin{bmatrix}\sigma^{-2}B^T(\rho_k(t)) & 0\end{bmatrix}\left\{\Pi(t,0)\begin{bmatrix}-\Pi_{11}^{-1}(\tau_k,0) & 0\\ 0 & 0\end{bmatrix}\Pi(\tau_k,s)\right. \\ &\left.\begin{bmatrix}0\\ \sigma^{-2}B(\rho_k(s))\end{bmatrix} + 1(t-s)\Pi(t,s)\begin{bmatrix}0\\ \sigma^{-2}B(\rho_k(s))\end{bmatrix}\right\} + \sigma^{-2}I\delta(t-s) \end{aligned}$$

From (13) the kernel of $\mathfrak{B}^*$ is $\mathfrak{B}^T(\rho_k(s))\Phi_k^T(\tau_k,s)$. Next, to determine kernel of $(\sigma^2 I - \mathfrak{D}_k^*\mathfrak{D}_k)^{-1}\mathfrak{B}_k^*$, we have

$$\begin{aligned} &\left((\sigma^2 I - \mathfrak{D}_k^*\mathfrak{D}_k)^{-1}\mathfrak{B}_k^*\right)(t) = \\ &\quad \begin{bmatrix}\sigma^{-2}B^T(\rho_k(t)) & 0\end{bmatrix}\left\{\Pi(t,0)\begin{bmatrix}-\Pi_{11}^{-1}(\tau_k,0) & 0\\ 0 & 0\end{bmatrix}\right. \\ &\quad + \int_0^{\tau_k}\Pi(\tau_k,s)\begin{bmatrix}0\\ \sigma^{-2}B(\rho_k(s))B^T(\rho_k(s))\end{bmatrix}\Phi_k^T(\tau_k,s)ds \\ &\quad \left. + \int_0^t \Pi(t,s)\begin{bmatrix}0\\ \sigma^{-2}B(\rho_k(s))B^T(\rho_k(s))\end{bmatrix}\Phi_k^T(\tau_k,s)ds\right\} \\ &\quad + \sigma^{-2}B^T(\rho_k(t))\Phi_k^T(\tau_k,t) \end{aligned} \tag{25}$$

Notice that

$$\frac{d}{ds}\left\{\Pi(t_1,s)\begin{bmatrix}I\\ 0\end{bmatrix}\Phi_k^T(t_2,s)\right\} = -\Pi(t_1,s)\begin{bmatrix}0\\ \sigma^{-2}BB^T\end{bmatrix}\Phi_k^T(t_2,s)$$

Evaluating the integrals in (25), we have

$$\left((\sigma^2 I - \mathfrak{D}_k^*\mathfrak{D}_k)^{-1}\mathfrak{B}_k^*\right)(t) = \begin{bmatrix}\sigma^{-2}B^T(\rho_k(t)) & 0\end{bmatrix}\Pi(t,0)\begin{bmatrix}\Pi_{11}^{-1}(\tau_k,0)\\ 0\end{bmatrix}$$

Finally for $\mathfrak{B}_k(\sigma^2 I - \mathfrak{D}_k^*\mathfrak{D}_k)^{-1}\mathfrak{B}_k^*$, we have

$$\begin{aligned} &\mathfrak{B}_k(\sigma^2 I - \mathfrak{D}_k^*\mathfrak{D}_k)^{-1}\mathfrak{B}_k^* = \\ &\int_0^{\tau_k}\Phi_k(\tau_k,t)B(\rho_k(t))\left((\sigma^2 I - \mathfrak{D}_k^*\mathfrak{D}_k)^{-1}\mathfrak{B}_k^*\right)(t)dt = \\ &\left\{\int_0^{\tau_k}\Phi_k(\tau_k,t)\begin{bmatrix}\sigma^{-2}B(\rho_k(t))B^T(\rho_k(t)) & 0\end{bmatrix}\Pi(t,0)dt\right\} \\ &\begin{bmatrix}\Pi_{11}^{-1}(\tau_k,0)\\ 0\end{bmatrix} \end{aligned}$$

and the evaluation of the integrals yields

$$\begin{aligned} \mathfrak{B}_k(\sigma^2 I - \mathfrak{D}_k^*\mathfrak{D}_k)^{-1}\mathfrak{B}_k^* &= \begin{bmatrix}0 & I\end{bmatrix}\Pi(\tau_k,0)\begin{bmatrix}0\\ I\end{bmatrix}\Pi_{11}^{-1}(\tau_k,0) \\ &= \Pi_{21}(\tau_k,0)\Pi_{11}^{-1}(\tau_k,0) \end{aligned}$$

The other equations in (19) can be derived similarly.